<u>Warm-Up</u>: Show that the sentence

$$(P \wedge Q) \Rightarrow P$$

is always true, no matter what the sentences P and Q are.

Do this in 2 ways:

1. With a truth table.
2. Write a conditional proof.

This shows $(P \wedge Q) \Rightarrow P$ is a <u>tautology</u>.

A sentence is called a <u>tautology</u> if it is true regardless of the truth value of the sentences involved.

Note: To prove $A \Rightarrow C$ is true, it is enough to show it is impossible for A to be true and C to simultaneously be false.

To eliminate this possibility, either

- Assume A is true and show C must also be true,
  ↳ Conditional proof of $A \Rightarrow C$

OR

- Assume C is false and show A must also be false
  ↳ Conditional proof of $\neg C \Rightarrow \neg A$

More Tautologies

Ex: The "law of the excluded middle":

$P \vee \neg P$ is a tautology. (Why?)

Ex: "Modus ponens"

$$(P \Rightarrow Q) \wedge P \Rightarrow Q$$

is a tautology.

Proof: Assume $(P \Rightarrow Q) \wedge P$ is true.

[Again, if it's false there's nothing to do.]

Then both of $P \Rightarrow Q$ and P are true.

Since P is true and $P \Rightarrow Q$ is true, we conclude Q is true. ∎

Modus ponens is a common step in logical reasoning.

Ex: "If it is raining, then the ground is wet. It is raining. Therefore, we can conclude that the ground is wet."

Ex: Show

$$[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$$

is a tautology.

"Hypothetical syllogism"

Proof: Suppose $[(P \Rightarrow Q) \wedge (Q \Rightarrow R)]$ is true.

Then $P \Rightarrow Q$ and $Q \Rightarrow R$ are both true.

We must show $P \Rightarrow R$ is true.

Suppose P is true.

[if P is false, then $P \Rightarrow R$ is automatically true.]

Since $P \Rightarrow Q$ is also true, Q is true. [modus ponens]

Since $Q \Rightarrow R$ is also true, R is true [modus ponens]

Thus, $P \Rightarrow R$ is true.

$\blacksquare$

Quantifiers

Consider

① "If $x > 1$, then $x^2 > 1$."

② "If $x^2 > 1$, then $x > 1$."

① is true no matter what x is.

② can be true ($x = 5$, $x = 0$, ...) or false ($x = -5$, ...) depending on x.

To "eliminate" the variable, we can write

①′ "For every $x \in \mathbb{R}$, if $x > 1$ then $x^2 > 1$."

②′ "For every $x \in \mathbb{R}$, if $x^2 > 1$ then $x > 1$."

[$\mathbb{R}$ is the set of real numbers, $x \in \mathbb{R}$ means x is a real number.]

Now, 1' is true and 2' is false.

There is no more dependence on x.

Since 2' is false, its negation is true

$\neg$ 2' "There is (at least one) $x \in \mathbb{R}$ such that $x^2 > 1$ and $x \leq 1$."

Ex: The negation of "every day is Friday" is

"there is at least one day which is not Friday"

NOT

"every day is not Friday"