Warm-Up: What is the difference between

(a) $(\exists x \in \mathbb{R})(\cos x = 0 \text{ and } \tan x = 0)$

and

(b) $(\exists x \in \mathbb{R})(\cos x = 0)$ and $(\exists x \in \mathbb{R})(\tan x = 0)$?

Free + Bound Variables

Let $P(x) =$ "$x^2 + 6x + 8 \geq 0.$"

- Is $P(x)$ true? It depends on x.

  We say that x is a **free variable** in the sentence $P(x)$.

  Think: The sentence $P(x)$ is a function of x.

- Is $(\forall x \in \mathbb{R})\, P(x)$ true? No!

  This sentence does NOT depend on x, because of the quantifier $\forall$.

In this case, we say x is a <u>bound variable</u> in the sentence $(\forall x \in \mathbb{R})\, P(x)$.

The quantifier $\exists$ can also bound variables: $(\exists x)\, P(x)$ does not depend on x.

<u>Analogy</u>: $f(x) = x^2$ vs. $\int_0^1 x^2\, dx$

<u>Note</u>: When we use a quantifier ($\forall$ or $\exists$), a bound variable is ranging over a <u>universe</u> of possibilities.

Usually, we should be explicit about this.

<u>Common choices</u>:

$\mathbb{Z}$ = the set of integers
$\mathbb{Q}$ = the set of rational numbers
$\mathbb{R}$ = the set of real numbers
$\mathbb{C}$ = the set of complex numbers

The universe matters!

Ex: $(\exists x)(x^2=2)$ Ambiguous

$(\exists x\in\mathbb{Z})(x^2=2)$ False, $\sqrt{2}\notin\mathbb{Z}$

$(\exists x\in\mathbb{R})(x^2=2)$ True, $\sqrt{2}\in\mathbb{R}$

Ex: $(\forall x\in\mathbb{R})(x^2\geq 0)$ True

$(\forall x\in\mathbb{C})(x^2\geq 0)$ False, $\sqrt{-1}\in\mathbb{C}$

Note: Over a finite set (universe),

- $\forall$ is an "and" statement
- $\exists$ is an "or" statement

Ex: If $A=\{-3, 1, 4\}$, then

$$(\forall x\in A)(x^2<20) \equiv ((-3)^2<20)\wedge(1^2<20)\wedge(4^2<20)$$

$$(\exists x\in A)(x>0) \equiv (-3>0)\vee(1>0)\vee(4>0)$$

(Both true)

We might say $\forall$ is "generalized and"

$\exists$ is "generalized or"

Thm (Generalized DeMorgan's Laws)

(a) $\neg\left[(\forall x \in A)\ P(x)\right] \equiv (\exists x \in A)(\neg P(x))$

(b) $\neg\left[(\exists x \in A)\ P(x)\right] \equiv (\forall x \in A)(\neg P(x))$

Proof: (a) Suppose $\neg\left[(\forall x \in A)\ P(x)\right]$ is true.
Then $(\forall x \in A)\ P(x)$ is false.

So there is some $x_0 \in A$ such that $P(x_0)$ is false, i.e. $\neg P(x_0)$ is true.

Hence $(\exists x \in A)(\neg P(x))$ is true.

Conversely, suppose $(\exists x \in A)(\neg P(x))$ is true.

Then there is $x_0 \in A$ such that $\neg P(x_0)$ is true, i.e. $P(x_0)$ is false.

So $(\forall x \in A)\, P(x)$ is false. Therefore, $\neg(\forall x \in A)\, P(x)$ is true.

(b) is similar (see book). ∎

Thm (Generalized Distributive Laws):

Let P be a sentence not involving x.
Let $Q(x)$ be a sentence involving x.
Then

$$\text{a) } P \wedge \left[(\exists x \in A)\, Q(x)\right] \equiv (\exists x \in A)\left[P \wedge Q(x)\right]$$

$$\text{b) } P \vee \left[(\forall x \in A)\, Q(x)\right] \equiv (\forall x \in A)\left[P \vee Q(x)\right].$$

Proof: Omitted (see book).

Order of Quantifiers

Suppose $P(x,y)$ is a sentence involving 2 variables.
What is the difference between

(a) $(\forall x)\left[(\exists y)\, P(x,y)\right]$

and

(b) $(\exists y)\left[(\forall x)\, P(x,y)\right]$?

implicit

Ex: $P(x,y)$ = " $x+y=1$ "

(a) is " for any $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $x+y=1$ " True!

Proof: Let $x \in \mathbb{R}$. Set $y = 1-x$. Then $y \in \mathbb{R}$ and $x+y = x+(1-x) = 1$. ∎

(b) is " there is $y \in \mathbb{R}$ such that for any $x \in \mathbb{R}$, we have $x+y=1$ " False!

How to prove? Let's show $\neg$(b) is true.

By De Morgan,

$$\neg (\exists y)\left[(\forall x)\ P(x,y)\right] \equiv (\forall y)\ \neg\left[(\forall x)\ P(x,y)\right]$$
$$\equiv (\forall y)\left[(\exists x)\ \neg P(x,y)\right]$$

Proof: Let $y \in \mathbb{R}$. We must show there is $x \in \mathbb{R}$ such that $x+y \neq 1$. Take $x = -y$. Then $x+y = (-y)+y = 0 \neq 1$. ∎

$$\boxed{\begin{array}{ll} \text{(a)} & (\forall x)\left[(\exists y)\ P(x,y)\right] \\ \quad\text{and} & \\ \text{(b)} & (\exists y)\left[(\forall x)\ P(x,y)\right] \end{array}}$$

In (a), we choose y after we know x.

In (b), we choose y first, and it has to work with every x.