Warm-up: What is the difference between

(a) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x \leq y)$
(b) $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(x \leq y)$?

Is either true?

Thm: Let $P(x,y)$ be a sentence depending on $x \in A$ and $y \in B$. Then

$$(\exists y \in B)(\forall x \in A)\, P(x,y) \Rightarrow (\forall x \in A)(\exists y \in B)\, P(x,y).$$

Proof: Assume $(\exists y \in B)(\forall x \in A)\, P(x,y)$ is true.

Then there is some $y_0 \in B$ such that

$$(\forall x \in A)\, P(x, y_0) \text{ is true.}$$

That is, for each $x \in A$, $P(x, y_0)$ is true.

Then $(\exists y \in B) P(x,y)$ is true for each $x \in A$, because we can take $y = y_0$.

In other words, $(\forall x \in A)(\exists y \in B) P(x,y)$ is true. ∎

Quantifiers of the same type commute.

Thm: Let $P(x,y)$ be a sentence depending on $x \in A$ and $y \in B$. Then

(a) $(\forall x \in A)\left[(\forall y \in B)\, P(x,y)\right] \equiv (\forall y \in B)\left[(\forall x \in A)\, P(x,y)\right]$

(b) $(\exists x \in A)\left[(\exists y \in B)\, P(x,y)\right] \equiv (\exists y \in B)\left[(\exists x \in A)\, P(x,y)\right]$

Proof: Talk it out.

Unique Existence

$\exists!$ is the <u>unique existential quantifier</u>

$(\exists!\, x \in A)\, P(x)$ means "there exists a unique (i.e. one and only one) $x \in A$ such that $P(x)$ is true."

Ex: ① $(\exists!\, x \in \mathbb{R})\,(x^2 = 0)$

True. $x^2 = 0 \iff x = 0$.

② $(\exists!\, x \in \mathbb{R})(x^2 = 2)$

False. $x = \sqrt{2}$ and $x = -\sqrt{2}$ both satisfy $x^2 = 2$.

③ $(\exists!\, x \in \mathbb{R})(x^2 = -2)$

False. There does not exist any $x \in \mathbb{R}$ such that $x^2 = -2$.

④ $(\forall x \in \mathbb{R})[x \neq 0 \Rightarrow (\exists!\, y \in \mathbb{R})(xy = 1)]$

True. If $x \neq 0$, set $y = \frac{1}{x}$.

Observation: $\exists!$ can be written in terms of $\forall$ and $\exists$.

$(\exists!\, x \in A)\, P(x)$ means

$$(\exists x \in A)\left[P(x) \wedge \underbrace{(\forall y \in A)(P(y) \Rightarrow (x = y))}_{\text{Any other solution is the one we already know.}}\right]$$

Induction

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of <u>natural numbers</u>.

You might have seen the following formula in Calc 2:

If $n \in \mathbb{N}$, then

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

In Σ-notation, this is

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

This is really infinitely many statements, one for each $n \in \mathbb{N}$.

i.e. $(\forall n \in \mathbb{N})\left(\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\right)$.

<u>$n=1$:</u> $1 = \frac{1 \cdot 2}{2}$

<u>$n=2$:</u> $1+2 = \frac{2 \cdot 3}{2}$

<u>$n=3$:</u> $1+2+3 = \frac{3 \cdot 4}{2}$

<u>$n=4$:</u> $1+2+3+4 = \frac{4 \cdot 5}{2}$

$\vdots$

How do we prove infinitely many statements?