Homework 14

Math 3345 – Spring 2023 – Kutler

Exercises

Please complete the following problems on your own paper. Solutions should be written clearly, legibly, and with appropriate style.

1. Let $a, b \in \mathbb{N}$. We say that a positive integer $m \in \mathbb{N}$ is a **common multiple** of a and b if $a|m$ and $b|m$.
   - (a) Show that for any $a, b \in \mathbb{N}$, ab is a common multiple of a and b.
   - (b) Prove that for any $a, b \in \mathbb{N}$, there exists a common multiple ℓ of a and b such that $\ell \leq m$ if m is any common multiple of a and b. This number ℓ is called the **least common multiple** of a and b. We write $\ell = \operatorname{lcm}(a, b)$.
   - (c) Give an example of positive integers $a, b \in \mathbb{N}$ such that $\operatorname{lcm}(a, b) = ab$.
   - (d) Give an example of positive integers $a, b \in \mathbb{N}$ such that $\operatorname{lcm}(a, b) < ab$.
   - (e) Explain why there do not exist positive integers a and b such that $\operatorname{lcm}(a, b) > ab$.
2. [**Falkner Section 4 Exercise 25**] Let $m \in \mathbb{N}$. Show that
   - (a) For all $a \in \mathbb{Z}$, we have $a \equiv a \mod m$. [**Reflexivity**]
   - (b) For all $a, b \in \mathbb{Z}$, if $a \equiv b \mod m$, then $b \equiv a \mod m$. [**Symmetry**]
   - (c) For all $a, b, c \in \mathbb{Z}$, if $a \equiv b \mod m$ and $b \equiv c \mod m$, then $a \equiv c \mod m$. [**Transitivity**]
3. [**Falkner Section 4 Exercise 26 – modified**] Let $m \in \mathbb{N}$ and $a, b, c, d \in \mathbb{Z}$. Suppose that $a \equiv b \mod m$ and $c \equiv d \mod m$.
   - (a) Prove that $a + c \equiv b + d \mod m$.
   - (b) Prove that $a - c \equiv b - d \mod m$.
   - (c) Prove that $ac \equiv bd \mod m$. [HINT: Since $a \equiv b \mod m$, m divides $b - a$, so $b - a = mk$ for some integer k. Rewrite this as $b = a + mk$. Similarly, $d = c + m\ell$ for some integer ℓ.]

Practice Problems

It is strongly recommended that you complete the following problems. There is no need to write up polished, final versions of your solutions (although you may find this a useful exercise). Please do not submit any work for these problems.

1. Let $a, b \in \mathbb{N}$. Prove that $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab$.

2. **[Falkner Section 4 Exercise 16]** Let $n \in \mathbb{N}$. Prove that there exists a prime number q such that $n < q \leq 1 + n!$. [HINT: Take q to be any prime which divides $1 + n!$. (How do we know such a prime exists?) Now explain why $q \leq 1 + n!$ and $q > n$ must both be true.]