Homework 21

Math 3345 – Spring 2023 – Kutler

Exercises

Please complete the following problems on your own paper. Solutions should be written clearly, legibly, and with appropriate style.

1. [**Falkner Section 11 Exercise 6**] Let $f(x) = x^2 + 1$ for all $x \in \mathbb{R}$, let $g(y) = \sqrt{y-1}$ for all $y \in [1, \infty)$, and let $h(u) = 1 - u$ for all $u \in [2, 3)$. Find the range of f, the range of g, and the range of h.

2. [**Falkner Section 11 Exercise 15(a) – modified**] Recall that for a set X, the **power set** $\mathscr{P}(X)$ is the set of all subsets of X.

   Let S and T be sets.

   (a) Prove that if $A \subseteq S$ and $B \subseteq T$, then $A \cup B \subseteq S \cup T$.

   (b) Prove that *every* subset of $S \cup T$ is of the form $A \cup B$, where $A \subseteq S$ and $B \subseteq T$. That is, if $Y \subseteq S \cup T$, then there exist subsets $A \subseteq S$ and $B \subseteq T$ such that $Y = A \cup B$.

   We may understand the result of part (a) as saying that the function

   $$\begin{aligned} f\colon \mathscr{P}(S) \times \mathscr{P}(T) &\to \mathscr{P}(S \cup T) \\ (A, B) &\mapsto A \cup B \end{aligned}$$

   is well-defined. That is, if $A \subseteq S$ and $B \subseteq T$, then $f(A, B) = A \cup B$ is a well-defined subset of $S \cup T$.

   The result of part (b) then shows that the range of f is all of $\mathscr{P}(S \cup T)$, i.e., f is surjective. That is, if $Y \in \mathscr{P}(S \cup T)$, then there exist $A \in \mathscr{P}(S)$ and $B \in \mathscr{P}(T)$ such that $f(A, B) = Y$.

   (c) Illustrate this line of thinking in the case where $S = \{1, 2\}$ and $T = \{2, 3\}$. The eight subsets of $S \cup T = \{1, 2, 3\}$ are

   $$\varnothing,\ \{1\},\ \{2\},\ \{3\},\ \{1,2\},\ \{1,3\},\ \{2,3\},\ \{1,2,3\}.$$

   For each set Y in this list, find $A \subseteq \{1, 2\}$ and $B \subseteq \{2, 3\}$ such that $A \cup B = Y$.

   (d) Continuing with the example from part (c), for which of the sets Y is there a **unique** choice of $A \subseteq \{1, 2\}$ and $B \subseteq \{2, 3\}$ such that $A \cup B = Y$?

Practice Problems

It is strongly recommended that you complete the following problems. There is no need to write up polished, final versions of your solutions (although you may find this a useful exercise). Please do not submit any work for these problems.

1. [**Falkner Section 10 Exercise 34 – modified**]

   (a) Let A be a set. Prove that $A \times \varnothing = \varnothing$.

   (b) Let A and B be sets. Deduce that $A \times \varnothing = B \times \varnothing$.

   (c) Let A, B, and C be sets, and suppose that $C \neq \varnothing$. Prove that if $A \times C = B \times C$, then $A = B$.

2. [**Falkner Section 10 Exercise 26**] Prove Theorem 10.36(b): Let S be a set and let $\mathscr{A}$ be a nonempty set of sets. Then

$$S \cup \left(\bigcap_{A \in \mathscr{A}} A \right) = \bigcap_{A \in \mathscr{A}} (S \cup A).$$

3. [**Falkner Section 10 Exercise 27**] Let A be a set and let $\mathscr{B}$ be a nonempty set of sets. Show that:

   (a) $A \cup \left(\bigcup_{B \in \mathscr{B}} B \right) = \bigcup_{B \in \mathscr{A}} (A \cup B)$

   (b) $A \cap \left(\bigcap_{B \in \mathscr{B}} B \right) = \bigcap_{B \in \mathscr{A}} (A \cap B)$