# Homework 22

Math 3345 – Spring 2023 – Kutler

## Exercises

Please complete the following problems on your own paper. Solutions should be written clearly, legibly, and with appropriate style.

1. [**Falkner Section 11 Exercise 15(b) – modified**] Let $S$ and $T$ be sets. Prove that $S$ and $T$ are disjoint (i.e., $S \cap T = \varnothing$) if and only if the following condition holds:

   For all subsets $A_1, A_2 \subseteq S$ and $B_1, B_2 \subseteq T$, if $A_1 \cup B_1 = A_2 \cup B_2$, then $A_1 = A_2$ and $B_1 = B_2$. $(\star)$

   **Note:** The condition $(\star)$ is equivalent to the statement that the function

   $$\begin{aligned} f \colon \mathscr{P}(S) \times \mathscr{P}(T) &\to \mathscr{P}(S \cup T) \\ (A, B) &\mapsto A \cup B \end{aligned}$$

   is injective. You proved on Homework 21 that this function is always surjective.

2. [**Falkner Section 11 Exercise 26**] Let $A$, $B$, and $C$ be sets. Prove that if $f \colon A \to B$ and $g \colon B \to C$ are bijections, then $g \circ f \colon A \to C$ is a bijection.

3. [**Falkner Section 11 Exercise 17 – modified**] Let

   $$\begin{aligned} f \colon [1, \infty) &\to \mathbb{R} \\ x &\mapsto x - 1. \end{aligned}$$

   (a) Show that $\operatorname{Rng}(f) \subseteq [0, \infty)$. That is, $f(x) \in [0, \infty)$ for every $x \in [1, \infty)$.

   (b) Prove that $\operatorname{Rng}(f) = [0, \infty)$. [HINT: In light of part (a), you need only prove the other inclusion, $[0, \infty) \subseteq \operatorname{Rng}(f)$. That is, for each $y \in [0, \infty)$, you must find some $x \in \operatorname{Dom}(f) = [1, \infty)$ such that $f(x) = y$.]

   (c) Prove that $f$ is an injection.

   (d) Conclude that $f$ is a bijection from $[1, \infty)$ to $[0, \infty)$, and give a formula for the inverse function $f^{-1} \colon [0, \infty) \to [1, \infty)$.

   (e) Sketch the graph of $f$.

## Practice Problems

It is strongly recommended that you complete the following problems. There is no need to write up polished, final versions of your solutions (although you may find this a useful exercise). Please do not submit any work for these problems.

1. [**Falkner Section 10 Exercise 28**] Find $\mathscr{P}\big(\{1,2,3\}\big)$.
2. [**Falkner Section 10 Exercise 35**] Let $A$, $B$, $C$, and $D$ be sets. Suppose that $A \times B = C \times D \neq \varnothing$. Prove that $A = C$ and $B = D$.
3. [**Falkner Section 10 Exercise 36**] Let $A$, $B$, and $C$ be sets. Prove that
$$A \setminus C \subseteq (A \setminus B) \cup (B \setminus C).$$
4. [**Falkner Section 11 Exercise 7**] Let $A$ and $B$ be sets, and let $\pi_A$ and $\pi_B$ be the projections from $A \times B$ to $A$ and $B$ respectively. That is,
$$\begin{aligned} \pi_A \colon A \times B &\to A \\ (x,y) &\mapsto x. \end{aligned} \qquad\qquad \begin{aligned} \pi_B \colon A \times B &\to B \\ (x,y) &\mapsto y. \end{aligned}$$
Show that if $B \neq \varnothing$, then $\mathrm{Rng}(\pi_A) = A$, and that if $A \neq \varnothing$, then $\mathrm{Rng}(\pi_B) = B$.