Homework 23

Math 3345 – Spring 2023 – Kutler

Exercises

Please complete the following problems on your own paper. Solutions should be written clearly, legibly, and with appropriate style.

1. **[Falkner Section 11 Exercise 20 – modified]** Let

$$g\colon [0,1) \to \mathbb{R} \qquad\qquad h\colon (-1,0) \to \mathbb{R}$$
$$x \mapsto \frac{x}{1-x}. \qquad\qquad x \mapsto \frac{x}{1+x}.$$

   (a) Prove that $\mathrm{Rng}(g) = [0,\infty)$ and $\mathrm{Rng}(h) = (-\infty, 0)$.

   (b) Prove that both g and h are injections.

   (c) Conclude that g is a bijection from $[0,1)$ to $[0,\infty)$ and that h is a bijection from $(-1,0)$ to $(-\infty,0)$.

   (d) Find formulas for $g^{-1}\colon [0,\infty) \to [0,1)$ and $h^{-1}\colon (-\infty,0) \to (-1,0)$.

2. **[Falkner Section 15 Exercise 1 – modified]** Show that the intervals $A = [1,\infty)$ and $B = (1,\infty)$ have the same cardinality by giving an example of a bijection $f\colon A \to B$.

   [Hint: Use one simple formula to define f on $\mathbb{N}$ and a different, even simpler formula to define f on $A \setminus \mathbb{N}$.]

   Be sure to prove that f is a bijection.

Practice Problems

It is strongly recommended that you complete the following problems. There is no need to write up polished, final versions of your solutions (although you may find this a useful exercise). Please do not submit any work for these problems.

1. Let

$$A = \{n \in \mathbb{N} \mid n \equiv 3 \mod 4\}.$$

   Define a bijection

$$f\colon \mathbb{N} \to A$$

   and prove that it is a bijection.

2. **[Falkner Section 11 Exercise 23]** Let

$$\begin{aligned} \varphi\colon (-1,1) &\to \mathbb{R} \\ x &\mapsto \frac{x}{1-|x|}. \end{aligned}$$

   (a) Show that φ is a bijection from $(-1,1)$ to $\mathbb{R}$.

   (b) Find a formula for $\varphi^{-1}\colon \mathbb{R} \to (-1,1)$.

   [HINT: Use Exercice 1 above.]

3. **[Falkner Section 15 Exercises 6 & 7 – modified]**

   (a) Show that the intervals $[0,1)$ and $(0,1]$ have the same cardinality by giving an example of a bijection $f\colon [0,1) \to (0,1]$.

   (b) Show that the intervals $(0,1]$ and $(0,1)$ have the same cardinality by giving an example of a bijection $g\colon (0,1] \to (0,1)$.

   (c) Show that the intervals $[0,1]$ and $[0,1)$ have the same cardinality by giving an example of a bijection $h\colon [0,1] \to [0,1)$.

   (d) Conclude that the four intervals $[0,1]$, $[0,1)$, $(0,1]$, and $(0,1)$ all have the same cardinality.

   (e) Use the functions f, g, and h to construct a bijection from $[0,1]$ to $(0,1)$. [HINT: Use Exercise 2 from Homework 22.]