

EXAM 1 PRACTICE PROBLEMS

1. Show that $(P \wedge Q) \vee R$ is logically equivalent to $(P \vee R) \wedge (Q \vee R)$ in two ways
 - (a) By using a truth table;
 - (b) By giving an explanation in words.
2. Below, you are asked to determine if two sentences are logically equivalent. If the answer is **yes**, provide a proof (either in words or by using a truth table). If the answer is **no**, demonstrate this by choosing appropriate truth values for P , Q , R , and provide a brief justification.
 - (a) Is $P \Rightarrow (Q \Rightarrow R)$ logically equivalent to $(P \Rightarrow Q) \Rightarrow R$?
 - (b) Is $P \Rightarrow (Q \Rightarrow R)$ logically equivalent to $Q \Rightarrow (P \Rightarrow R)$?
 - (c) Is $(P \wedge Q) \Rightarrow R$ logically equivalent to $(P \Rightarrow R) \vee (Q \Rightarrow R)$?
 - (d) Is $(P \vee Q) \Rightarrow R$ logically equivalent to $(P \Rightarrow R) \wedge (Q \Rightarrow R)$?
 - (e) Is $\neg[(P \Rightarrow Q) \wedge P]$ logically equivalent to $(Q \Rightarrow P) \vee \neg P$?
3. For each sentence below, determine if the sentence is **always true** (i.e., it is a tautology) or if it is **possibly false**. If it is always true, provide a proof (either in words or by using a truth table). If it is possibly false, give truth values for P , Q , R making the sentence false, and provide a brief justification.
 - (a) $P \wedge \neg P$
 - (b) $P \vee \neg P$
 - (c) $(P \wedge Q) \Rightarrow (P \vee Q)$
 - (d) $(P \vee Q) \Rightarrow (P \wedge Q)$
 - (e) $P \Rightarrow (Q \Rightarrow P)$
 - (f) $(P \Rightarrow Q) \Rightarrow P$
 - (g) $[(P \Rightarrow Q) \wedge \neg Q] \Rightarrow \neg P$
 - (h) $[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$
4. In class, we saw that $[(P \Rightarrow Q) \wedge P] \Rightarrow Q$ is a tautology, called *modus ponens*.
 - (a) Show that $(P \Rightarrow Q) \wedge P$ is logically equivalent to $P \wedge Q$.
 - (b) Show that the converse of modus ponens is not a tautology. That is, find truth values for P and Q so that the sentence $Q \Rightarrow [(P \Rightarrow Q) \wedge P]$ is false.

5. Show that each of the following conditional sentences is a tautology by writing a conditional proof.

(a) $P \Rightarrow (P \vee Q)$

(b) $[P \Rightarrow (Q \wedge \neg Q)] \Rightarrow \neg P$

(c) $[(P \Rightarrow \neg Q) \wedge (R \Rightarrow Q)] \Rightarrow (P \Rightarrow \neg R)$

(d) $\{[(P \Rightarrow Q) \wedge (R \Rightarrow S)] \wedge (\neg Q \vee \neg S)\} \Rightarrow (\neg P \vee \neg R)$

6. Below, you are asked to determine if two sentences are logically equivalent. If the answer is **yes**, provide a proof. If the answer is **no**, demonstrate this by giving a set A and component sentences $P(x)$ and $Q(x)$ making it false, and provide a brief justification.

(a) Is

$$(\exists x \in A)(P(x) \wedge Q(x))$$

logically equivalent to

$$((\exists x \in A)P(x)) \wedge ((\exists x \in A)Q(x))?$$

(b) Is

$$(\forall x \in A)(P(x) \wedge Q(x))$$

logically equivalent to

$$((\forall x \in A)P(x)) \wedge ((\forall x \in A)Q(x))?$$

(c) Is

$$(\exists x \in A)(P(x) \Rightarrow Q(x))$$

logically equivalent to

$$((\exists x \in A)P(x)) \Rightarrow ((\exists x \in A)Q(x))?$$

7. For each of the following sentences, write out what it means in words, state whether it is true or false, and prove your statement.

(a) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(xy = 0)$

(b) $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(xy = 0)$

(c) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(xy = 20)$

(d) $(\forall x \in \mathbb{R})[(x \neq 0) \Rightarrow (\exists y \in \mathbb{R})(xy = 20)]$

(e) $(\forall x \in \mathbb{R})[(x \neq 0) \Rightarrow (\exists!y \in \mathbb{R})(xy = 20)]$

(f) $(\forall m \in \mathbb{Z})[(m \neq 0) \Rightarrow (\exists!n \in \mathbb{Z})(mn = 20)]$

(g) $(\forall m \in \mathbb{Z})(\exists n \in \mathbb{Z})(m < n)$

(h) $(\exists n \in \mathbb{Z})(\forall m \in \mathbb{Z})(m < n)$

8. Prove the following statements using mathematical induction. Be sure to clearly state the inductive hypothesis, and explain what you are proving in the inductive step.

(a) For every $n \in \mathbb{N}$,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

(b) For every $n \in \mathbb{N}$,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

(c) For every $n \in \mathbb{N}$,

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

(d) For every $n \in \mathbb{N}$,

$$1 + 3 + 5 + \cdots + (2n-1) = n^2.$$

(e) For every $n \in \mathbb{N}$ such that $n > 3$, $n! > 2^n$.

(f) For every $n \in \mathbb{N}$ such that $n > 6$, $n! > 3^n$.