Exam 2 Practice Problems

1. Prove the following.
   (a) The sum of two odd integers is even.
   (b) The sum of an even and an odd integer is odd.
   (c) The sum of two even integers is even.
   (d) The product of two odd integers is odd.
   (e) The product of an even integer and an odd integer is even.
   (f) The product of two even integers is even.

2. Let $n, m \in \mathbb{Z}$. Prove the following.
   (a) If nm is odd, then n is odd and m is odd.
   (b) If nm is even, then n is even or m is even.
   (c) If n^2 is odd, then n is odd.
   (d) If n^2 is even, then n is even.

3. Let $d, n \in \mathbb{N}$. Use the definition of divisibility to show that if $d|n$, then $d \leq n$.

4. Let $a, b \in \mathbb{Z}$. Use the definition of divisibility to show that if $a|b$, then $a^2|b^2$.

5. Let a, b, q, r be integers such that $a = bq + r$. Prove that $\gcd(a, b) = \gcd(b, r)$.

6. Let $d \in \mathbb{N}$ and $n \in \mathbb{Z}$. Show that if $d|n$ and $d|(n + 1)$, then $d = 1$.

7. Use the Euclidean algorithm to compute $\gcd(84, 135)$.

8. (a) Use the Euclidean algorithm to compute $\gcd(30, 72)$.
   (b) Find integers $x, y \in \mathbb{Z}$ such that $30x + 72y = 6$.
   (c) Do there exist integers $x, y \in \mathbb{Z}$ such that $30x + 72y = 48$?
   (d) Do there exist integers $x, y \in \mathbb{Z}$ such that $30x + 72y = 16$?

9. Find integers x and y such that $162x + 31y = 1$.

10. Let n be an even integer. Prove that there exist unique integers $q, r \in \mathbb{Z}$ such that

$$n = 6q + r$$

and $r \in \{0, 2, 4\}$.

11. Make addition and multiplication tables for arithmetic
    (a) modulo 2.
    (b) modulo 3.
    (c) modulo 4.
    (d) modulo 5.
12. Without using a calculator, find the natural number k such that $0 \leq k \leq 14$ and k satisfies the given congruence.
    (a) $2^{75} \equiv k \mod 15$
    (b) $6^{41} \equiv k \mod 15$
    (c) $140^{874} \equiv k \mod 15$
13. Without using a calculator, show that 15 divides $37^{42} - 38^{90}$.
14. (a) Check that $r^3 \equiv r \mod 6$ for every integer r such that $0 \leq r \leq 5$.
    (b) Use part (a) to prove that $n^3 \equiv n \mod 6$ for every integer n.
    (c) If x is a real number such that $x^3 = x$, then either $x = 0$ or we can divide by x to get $x^2 = 1$ (from which we conclude $x = 1$ or $x = -1$).

        Given the result of part (b), we might wonder if similar reasoning implies that for every integer n, either $n \equiv 0 \mod 6$ or $n^2 \equiv 1 \mod 6$. Is this true?
15. Use induction to prove that
$$7^n \equiv 1 + 6n \mod 9$$
    for every $n \in \mathbb{N}$.
16. (a) Let $x \in \mathbb{Z}$ and let p be a prime number. Prove that if p does not divide x, then $\gcd(p, x) = 1$.
    (b) Show that there exists $x \in \mathbb{Z}$ such that 12 does not divide x and $\gcd(12, x) \neq 1$. Why does this not contradict the result of part (a)?
17. Let P be the sentence

    For all $a, b \in \mathbb{Z}$, if $a|b$ then $a|(b + 5a^2)$.

    Let Q be the sentence

    For all $a, b \in \mathbb{Z}$, if $a|b$ then $b + 5a^2$ is not prime.

    (a) Is the sentence P true? If so, provide a proof. If not, provide a counterexample.
    (b) Is the sentence Q true? If so, provide a proof. If not, provide a counterexample.

18. (a) Fill in the blanks: According to the division algorithm, when we divide an integer n by 5, we obtain unique integers $q, r \in \mathbb{Z}$ such that

$$n = \underline{\hspace{3cm}}$$

and

$$\underline{\hspace{1cm}} \leq r \leq \underline{\hspace{1cm}}.$$

(b) Use the statement in part (a) to prove the following: For any integer $a \in \mathbb{Z}$, if $5|a^2$, then $5|a$.

[HINT: Apply part (a) to $n = a^2$ and to $n = a$.]

(c) Give an example of an integer a such that $6|a^2$ but $6\nmid a$.

(d) According to part (b), the implication

$$5|a^2 \;\Rightarrow\; 5|a \tag{$\star$}$$

is **true** for every $a \in \mathbb{Z}$. But part (c) shows that the similar implication

$$6|a^2 \;\Rightarrow\; 6|a \tag{$\star\star$}$$

is **false**, at least for some integers a.

We might try to adapt the proof of $(\star)$ from part (b) into a similar proof of $(\star\star)$. Of course, this necessarily will fail, because $(\star\star)$ is false. What, specifically, goes wrong when you do this?

19. Use the prime factorizations

$$3{,}219{,}398 = 2 \cdot 7^3 \cdot 13 \cdot 19^2 \qquad \text{and} \qquad 158{,}184 = 2^3 \cdot 3^2 \cdot 13^3$$

to find gcd(3,219,398 , 158,184). Explain your reasoning.

20. Let $a \in \mathbb{N}$ and let p be a prime number. Prove that if $p|a^2$, then $p|a$.

[HINT: Consider the prime factorizations of a and a^2.]

21. Give examples to prove the following statements.

(a) There exist irrational numbers x and y such that $x + y$ is irrational.

(b) There exist irrational numbers x and y such that $x + y$ is rational.

(c) There exist irrational numbers x and y such that xy is irrational.

(d) There exist irrational numbers x and y such that xy is rational.

22. Let $x, y \in \mathbb{R}$. Prove the following.

    (a) If x and y are rational, then $x + y$ is rational.
    (b) If x and y are rational, then xy is rational.
    (c) If y is rational and $y \neq 0$, then $1/y$ is rational.
    (d) If x and y are rational and $y \neq 0$, then x/y is rational.
    (e) If x is rational and y is irrational, then $x + y$ is irrational.
    (f) If $x \neq 0$ is rational and y is irrational, then xy is irrational.
    (g) If y is irrational, then $1/y$ is irrational. (Why must $y \neq 0$ be true?)
    (h) If $x \neq 0$ is rational and y is irrational, then x/y is irrational.

23. Prove the following.

    [HINT: Use the fact that any rational number can be written in lowest terms.]

    (a) $\sqrt{2}$ is irrational.
    (b) $\sqrt{3}$ is irrational.
    (c) $\sqrt{6}$ is irrational.
    (d) $\sqrt{2} + \sqrt{3}$ is irrational.

24. Let's prove that $\sqrt{24}$ is irrational.

    Suppose that $a^2 = 24b^2$ (equivalently, $\left(\frac{a}{b}\right)^2 = 24$) for some $a, b \in \mathbb{N}$. We will derive a contradiction.

    (a) By the Fundamental Theorem of Arithmetic, a has a unique prime factorization. Write it as

    $$a = p_1^{e_1} \cdots p_k^{e_k}, \tag{$\star$}$$

    where $p_1, \ldots, p_k$ are distinct primes (i.e., $p_i \neq p_j$ when $i \neq j$) and the exponents e_i are positive integers.

    Use ($\star$) to write the unique prime factorization of a^2.

    (b) Describe the unique prime factorization of $24b^2$.

    [HINT: Similar to part (a), start by writing the unique prime factorization of b.]

    (c) Use the equality $a^2 = 24b^2$ and your prime factorizations from parts (a) and (b) to get a contradiction. Conclude that no such $a, b \in \mathbb{N}$ exist.