__Warm-Up__: Let $x, y \in \mathbb{Z}$. Prove that if $x+y$ is odd, then x is odd or y is odd.

__Thm__: Every integer is even or odd.

__Proof__: Let $P(n)$ be "n is even or n is odd."

We will first prove $(\forall n \in \mathbb{N})\, P(n)$ by induction.

__Base Case__: When $n=1$, we have

$$1 = 2(0) + 1$$

proving that 1 is odd. So $P(1)$ is true.

__Inductive Step__: Let $n \in \mathbb{N}$ and assume $P(n)$ is true. That is, n is even or n is odd.

Case 1: n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$. Thus,

$$n+1 = 2k+1$$

is odd, proving $P(n+1)$ is true.

Case 2: n is odd. Then $n = 2\ell+1$ for some $\ell \in \mathbb{Z}$. Thus,

$$\begin{aligned} n+1 = (2\ell+1)+1 &= 2\ell+2 \\ &= 2(\ell+1). \end{aligned}$$

Since $\ell+1 \in \mathbb{Z}$, this shows $n+1$ is even. So $P(n+1)$ is true.

Thus, $P(n+1)$ is true in both cases. This completes the inductive step.

We conclude that $P(n)$ is true for each $n \in \mathbb{N}$.

It remains to prove $P(n)$ for $n \leq 0$.

Zero: $0 = 2(0)$ is even, so $P(0)$ is true.

Negatives: Every negative integer is of the form $-n$, where $n \in \mathbb{N}$. Thus, it suffices to prove

$$P(n) \Rightarrow P(-n)$$

for each $n \in \mathbb{N}$.

So assume $P(n)$ is true.

Case 1: n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$. Thus,

$$-n = -2k = 2(-k)$$

is even, since $-k \in \mathbb{Z}$.

Case 2: n is odd. Then $n = 2\ell + 1$ for some $\ell \in \mathbb{Z}$. Now,

$$\begin{aligned} -n = -(2\ell + 1) &= -2\ell - 1 \\ &= 2(-\ell - 1) + 1. \end{aligned}$$

Since $-\ell - 1 \in \mathbb{Z}$, this shows $-n$ is odd

Thus, $P(-n)$ is true in both cases.

Since $P(n)$ and $P(n) \Rightarrow P(-n)$ are both true for every $n \in \mathbb{N}$, we conclude $P(-n)$ is true for every $n \in \mathbb{N}$.

$\blacksquare$

Axioms for the integers

Axioms 1 – 10 on handout

- Every fact you know (or don't) about integers follows from these axioms.
- For the moment, let's imagine that we <u>only</u> know these axioms.

What can we deduce?

For example, it's not even clear that $\mathbb{N}$ is equal to $\{1, 2, 3, \ldots\}$.