Positive Integers

Goal: Use Axioms 7-10 to show

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

Axiom 7 simply tells us that $\mathbb{N}$ exists.

↳ Define $a < b$ to mean $b - a \in \mathbb{N}$.

Axiom 8 tells us that $\mathbb{N}$ is closed under $+$ and $\cdot$

Axiom 9 tells us there is a trichotomy:

Each $a \in \mathbb{Z}$ satisfies exactly one of

- $a \in \mathbb{N} \iff a > 0$ "a is positive"
- $a = 0$
- $-a \in \mathbb{N} \iff a < 0$ "a is negative"

Axiom 10 is mysterious...

Lemma 6: $1 \in \mathbb{N}$.

Proof: By trichotomy, we only need to eliminate the other two possibilities.

- $1 \neq 0$ by Axiom 5 (Identity)
- To show $-1 \in \mathbb{N}$ is false, we will assume it is true and derive a contradiction.

Suppose $-1 \in \mathbb{N}$. Then by Axiom 8,

$$(-1)\cdot(-1) = 1 \in \mathbb{N}$$

also. But $-1 \in \mathbb{N}$ and $1 \in \mathbb{N}$ cannot both be true, by Axiom 9.

Thus, $-1 \in \mathbb{N}$ is false.

The only remaining possibility is $1 \in \mathbb{N}$. $\blacksquare$

Note: We proved that $(-1 \in \mathbb{N})$ is false by contradiction.

In general, we can prove that a sentence P is false as follows:

① Assume P is true.

② Show that this assumption leads us to a contradiction. That is, we are forced to conclude that a sentence Q is true, even though we already know Q to be false.

Formally, if we prove

$$P \Rightarrow Q,$$

where Q is known to be false, then P must also be false.

<u>Lemma 7</u>: For any $a, b \in \mathbb{Z}$, if $a \cdot b = 0$, then $a = 0$ or $b = 0$.

<u>Proof</u>: Homework 9.

<u>Note</u>: Prove the contrapositive:

If $a \neq 0$ and $b \neq 0$, then $a \cdot b \neq 0$.

By trichotomy, $x \neq 0$ if and only if

- $x \in \mathbb{N}$ (i.e. $x > 0$)

or

- $-x \in \mathbb{N}$ (i.e. $x < 0$).

Now, consider cases.

Let's use this to prove:

Thm 10: For any $a, b, c \in \mathbb{Z}$ with $a \neq 0$, if $a \cdot b = a \cdot c$, then $b = c$.

[Multiplicative Cancellation]

Proof: Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$. Suppose $a \cdot b = a \cdot c$. Then

$$a \cdot b - a \cdot c = 0$$
$$a \cdot (b - c) = 0.$$

By the Lemma, $a = 0$ or $b - c = 0$. But $a \neq 0$, so $b - c = 0$.

That is, $b = c$. $\blacksquare$

Note: No division required!

(And no division <u>defined</u> in $\mathbb{Z}$.)

Lemma 9: For any $a, b \in \mathbb{Z}$, exactly one of the following is true:

- $a < b$
- $a = b$
- $a > b$

[That is, $\mathbb{Z}$ is <u>linearly ordered</u> by $<$]

Proof idea: Apply trichotomy to $b-a$.

Lemma 10: Let $a, b, c \in \mathbb{Z}$.

① If $a < b$, then $a+c < b+c$
② If $a < b$ and $c > 0$, then $a \cdot c > b \cdot c$.

Proof: ① Suppose $a < b$. Then $b-a \in \mathbb{N}$. Now,

$$(b+c)-(a+c) = b-a \in \mathbb{N},$$

so $a+c < b+c$.

② Suppose $a<b$ and $c>0$.
Then $b-a \in \mathbb{N}$ and $c \in \mathbb{N}$, so

$$(b-a)\cdot c \in \mathbb{N}$$

by Axiom 8. But

$$(b-a)\cdot c = b\cdot c - a\cdot c, \quad \text{(Axiom 4)}$$

so $a\cdot c < b\cdot c$. ∎

The Well-Ordering axiom (#10) is the only one we haven't used yet. It says that any non-empty subset of $\mathbb{N}$ has a <u>smallest element</u>.

An element $a \in S$ is the <u>smallest element</u> of S if for all $x \in S$, $a \le x$.

In symbols: $(\forall x \in S)(a \le x)$

Observe that a smallest element in S, if it exists, is unique.

$$(\forall x \in S)(a \le x) \text{ and } (\forall x \in S)(b \le x)$$

implies $a \le b$ and $b \le a$, so $a = b$.

Lemma: The integer 1 is the smallest element of $\mathbb{N}$.

Proof: First, we know $\mathbb{N}$ has a smallest element by the Well-Ordering axiom. Call it a.

Since $a \leq n$ for every $n \in \mathbb{N}$, we have $a \leq 1$. Therefore $a = 1$ or $a < 1$. If $a = 1$, we are done.

To show $a < 1$ is false, we will assume it's true and derive a contradiction.

So assume $a < 1$. Because $a \in \mathbb{N}$, $0 < a$. Multiply the inequality $a < 1$ by $a > 0$ to get

$$a \cdot a < 1 \cdot a = a.$$

Now, $a \cdot a \in \mathbb{N}$ by Positive Closure, which contradicts a being the smallest element of $\mathbb{N}$.

Thus, our assumption that $a < 1$ is false, so $a = 1$. ∎