Last time: 1 is the smallest positive integer.

This actually shows that the integers are what you think they are:

- We know $0, 1 \in \mathbb{Z}$ by the Identity axiom.

- We also know
$$\begin{aligned} 1+1 &= 2 \\ 2+1 &= 3 \\ 3+1 &= 4 \\ &\vdots \end{aligned}$$
are positive integers.

If there was another positive number x not on this list, then it would satisfy $n < x < n+1$ for some $n \in \mathbb{N}$. But then $0 < x-n < 1$, which violates the theorem.

So $\mathbb{N} = \{1, 2, 3, 4, \dots\}$.

- The only other integers, by Trichotomy, satisfy $-a \in \mathbb{N}$. That is, they are the additive inverses of elements in $\mathbb{N}$.

Together, $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

Note: The handout shows this slightly more rigorously, by proving the Principle of Mathematical Induction.

Divisibility

Def: Let d and n be integers. We say <u>d divides n</u> if there exists an integer k such that $n = dk$.

Note on definitions: A definition is a $\Leftrightarrow$ statement, but it is often written as a $\Rightarrow$ statement.

So

$$d \text{ divides } n \iff (\exists k \in \mathbb{Z})(n = dk)$$

Notation: $d \mid n$ means "d divides n"

Ex: $2 \mid n \iff n = 2k$ for some $k \in \mathbb{Z}$

$\iff n$ is even.

Ex: $3 \mid n \iff n = 3k$ for some $k \in \mathbb{Z}$

So 3 divides 3 $(3 = 3 \cdot 1)$

" " 9 $(9 = 3 \cdot 3)$

" " -6 $(-6 = 3 \cdot (-2))$

" " 0 $(0 = 3 \cdot 0)$

Def: When $d \mid n$, we say d is a <u>divisor</u> of n and n is a <u>multiple</u> of d.

Ex:
- Every integer d divides 0, because $0 = d \cdot 0$.
- 1 divides every integer n, because $n = 1 \cdot n$.
- 0 only divides itself, because $n = 0 \cdot k \Rightarrow n = 0$.

Ex: The divisors of 15 are $\pm 1, \pm 3, \pm 5, \pm 15$.

Warning: $d \mid n$ is the sentence "d divides n"
d/n is the number $\frac{d}{n}$

Note: When $d \neq 0$,

$$d \mid n \text{ is true} \iff n = d \cdot k \text{ for some } k \in \mathbb{Z}$$
$$\iff \frac{n}{d} \text{ is an integer}$$

We usually avoid division, as that can take us out of the integers.

Thm: Let $d, n \in \mathbb{Z}$. If $d \mid n$, then $(-d) \mid n$.

Proof: Suppose $d \mid n$. Then there exists $k \in \mathbb{Z}$ such that $n = dk$. Then

$$n = [(-1)\cdot(-1)] \cdot dk = (-d)(-k)$$

Since $-k \in \mathbb{Z}$, this shows $(-d) \mid n$. ∎

For this reason, we often only list positive divisors.

Thm: Let $d, n \in \mathbb{N}$. If $d \mid n$, then $d \leq n$.

Proof: Suppose $d \mid n$. Then there exists $k \in \mathbb{Z}$ such that

$$n = dk.$$

Now, $k \leq 0$ or $k \geq 1$.

Suppose, for the sake of contradiction, that $k \leq 0$. Since $d > 0$, $n = dk \leq 0$, which contradicts $n \in \mathbb{N}$.

So $k \geq 1$. Multiply by d to get

$$dk \geq d$$

i.e. $n \geq d$. $\blacksquare$