Last time: $\sqrt{2}$ is irrational.

Here is a vast generalization:

$$\boxed{\begin{array}{l}\text{For each } n \in \mathbb{N}, \text{ either} \\ \quad \bullet\ \sqrt{n} \in \mathbb{N} \\ \text{or} \\ \quad \bullet\ \sqrt{n} \text{ is irrational.}\end{array}}$$

This follows immediately from the following

Thm (Example 4.52 in text):
If $x \in \mathbb{Q}$ and $x^2 \in \mathbb{Z}$, then $x \in \mathbb{Z}$.

↳ Take $x = \sqrt{n}$ to prove the boxed statement.

Proof: Suppose $x \in \mathbb{Q}$ and $x^2 \in \mathbb{Z}$.

Write $x = \frac{a}{b}$ in lowest terms with $a \in \mathbb{Z}$ and $b \in \mathbb{N}$.

Our goal is to show $b=1$, so $x=a\in\mathbb{Z}$.

Let's assume $b\neq 1$ and get a contradiction.

Since $b\neq 1$ and $b\in\mathbb{N}$, we have $b>1$. Thus, there is some prime p such that $p\mid b$.

Now, $x^2=\frac{a^2}{b^2}=n$ for some $n\in\mathbb{Z}$. Thus,

$$a^2=b^2n=b(bn).$$

That is, $b\mid a^2$. By transitivity of divisibility, $p\mid a^2$ also.

But then $p\mid a$ by the "Theorem on Division by a Prime."

So $p\mid a$ and $p\mid b$, contradicting the fact that $\frac{a}{b}$ is in lowest terms.

Therefore, we conclude $b=1$ and thus $x\in\mathbb{Z}$. ∎

Sets

"**Def**": A <u>set</u> is an unordered collection of objects, called <u>elements</u> of the set.

Actual definition is a list of axioms

One way to describe a set: list its elements inside braces.

<u>Ex</u>: $\{1,2,3\}$, $\{\text{red}, \text{blue}\}$, $\{☺, \$, ★, □\}$ are sets

<u>Important notes</u>:

- The elements in a set are <u>unordered</u>.
  So
  $$\{1,2,3\},\ \{1,3,2\},\ \{2,1,3\},\ \{2,3,1\},\ \{3,1,2\},\ \{3,2,1\}$$
  are six ways of writing <u>the same set</u>.

- The elements are <u>distinct</u> – no object can appear more than once. If we write
  $$\{1,1,1,2,2,2,2,2,2,3,3\},$$
  this means the set $\{1,2,3\}$.

<u>Notation</u>: If A is a set, then $x \in A$ means x is an element of A. $x \notin A$ means x is not an element of A.

<u>Ex</u>: $A = \{1, 2, 3\}$. Then $2 \in A$ and ☺ $\notin A$.

Sets can have sets as elements.

<u>Ex</u>: $A = \{ \{1,2\}, \{\text{red}, \text{blue}\}, \$ \}$ is a set with three elements, two of which are sets themselves.

$$\{1,2\} \in A$$

$$1 \notin A$$