Warm-Up: List all subsets of

- $\{1\}$
- $\{1,2\}$
- $\{1,2,3\}$

Ex: Let $n \in \mathbb{N}$. A set with n elements has exactly 2^n subsets. Why?

Thm: ① For all sets A, $A \subseteq A$. [Reflexive]

② For all sets A and B, if $A \subseteq B$ and $B \subseteq A$, then $A = B$. [Antisymmetric]

③ For all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. [Transitive]

Note: $\leq$ and divisibility have these same 3 properties!

Proof: ① we proved last time.

② is our definition of set equality.

③: Suppose $A \subseteq B$ and $B \subseteq C$. This means

$$x \in A \Rightarrow x \in B \quad \text{is true for every } x$$

and

$$x \in B \Rightarrow x \in C \quad \text{is true for every } x.$$

To prove $A \subseteq C$, suppose $x \in A$ for some x. Then $x \in B$ because $A \subseteq B$. Thus, $x \in C$ because $B \subseteq C$.

Therefore, $x \in A \Rightarrow x \in C$ for every x, so $A \subseteq C$.

∎

Algebra of Sets

Def: Let A and B be sets.

① The <u>union</u> of A and B is the set

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

② The <u>intersection</u> of A and B is the set

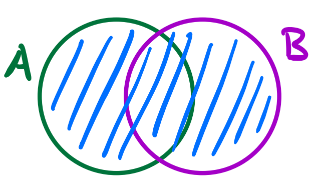
$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

③ The <u>relative complement</u> of B in A is the set

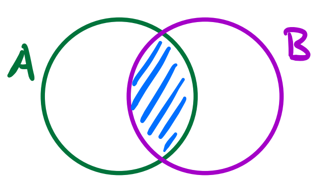
$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}.$$

Also called <u>set difference</u>

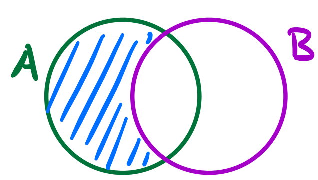
Pictures:



$A \cup B$



$A \cap B$



$A \setminus B$

<u>Ex</u>: Let $E = \{n \in \mathbb{N} \mid n \text{ is even}\}$
$= \{2, 4, 6, 8, \ldots\}$

and

$P = \{p \in \mathbb{N} \mid p \text{ is prime}\}$
$= \{2, 3, 5, 7, 11, \ldots\}$

- $E \cup P = \{n \in \mathbb{N} \mid n \text{ is even or prime}\}$
  $= \{2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 17, \ldots\}$
- $E \cap P = \{n \in \mathbb{N} \mid n \text{ is even and prime}\}$
  $= \{2\}$
- $E \setminus P = \{4, 6, 8, 10, \ldots\}$
- $P \setminus E = \{3, 5, 7, 11, \ldots\}$
- $\mathbb{N} \setminus E = \{n \in \mathbb{N} \mid n \text{ is odd}\}$
  $= \{1, 3, 5, 7, \ldots\}$
- $E \setminus \mathbb{N} = \emptyset$

Many theorems from logic translate directly to theorems about sets.

Lemma: Let A and B be sets. Then for any object x,

(a) $x \notin A \cup B$ if and only if $x \notin A$ and $x \notin B$.

(b) $x \notin A \cap B$ if and only if $x \notin A$ or $x \notin B$.

Proof: (a)

$$
\begin{aligned}
x \notin A \cup B &\iff \neg(x \in A \cup B) \\
&\iff \neg\left[(x \in A) \vee (x \in B)\right] \\
&\iff \neg(x \in A) \wedge \neg(x \in B) \qquad \text{DeMorgan} \\
&\iff (x \notin A) \wedge (x \notin B).
\end{aligned}
$$

(b) is similar, using the other DeMorgan Law. $\blacksquare$

Ex: $A \cap B \subseteq A$, $A \cap B \subseteq B$ (HW 18)

<u>Thm</u> (DeMorgan Laws for sets):

Let A, B, and S be sets. Then

(i) $S \setminus (A \cup B) = (S \setminus A) \cap (S \setminus B)$.

(ii) $S \setminus (A \cap B) = (S \setminus A) \cup (S \setminus B)$.

<u>Proof</u>: (i) We'll show both containments.

($\subseteq$): Let $x \in S \setminus (A \cup B)$. Then $x \in S$ and $x \notin A \cup B$. By the Lemma, $x \notin A$ and $x \notin B$. So $x \in S \setminus A$ and $x \in S \setminus B$. Thus, $x \in (S \setminus A) \cap (S \setminus B)$.

($\supseteq$): Let $x \in (S \setminus A) \cap (S \setminus B)$. Then $x \in S \setminus A$ and $x \in S \setminus B$. So $x \in S$ and $x \notin A$, and $x \in S$ and $x \notin B$. Since $x \notin A$ and $x \notin B$, we have $x \notin A \cup B$ by the Lemma. Thus, because $x \in S$, we have $x \in S \setminus (A \cup B)$.

(ii) is similar.

$\blacksquare$

Similarly, one can prove the following.

Thm (Commutativity of $\cup$ and $\cap$):
Let A and B be sets. Then

(i) $A \cup B = B \cup A$

(ii) $A \cap B = B \cap A$.

Thm (Associativity of $\cup$ and $\cap$):
Let A, B, and C be sets. Then

(i) $(A \cup B) \cup C = A \cup (B \cup C)$

(ii) $(A \cap B) \cap C = A \cap (B \cap C)$.

Thm (Distributive Laws for sets):
Let A, B, and S be sets. Then

(i) $S \cap (A \cup B) = (S \cap A) \cup (S \cap B)$

(ii) $S \cup (A \cap B) = (S \cup A) \cap (S \cup B)$