Warm-Up: Let A and B be sets. Show that

$$A \subseteq A \cup B \quad \text{and} \quad B \subseteq A \cup B.$$

HW 18: You showed $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

Recall:
$$x \in A \cup B \iff (x \in A) \vee (x \in B)$$
$$x \in A \cap B \iff (x \in A) \wedge (x \in B)$$

Sets of sets

Notation: We'll often use a script letter to denote a set of sets - i.e. a set, all of whose elements are sets.

Def: Let $\mathcal{A}$ be a set of sets. Then

- $\bigcup_{A \in \mathcal{A}} A = \{x \mid (\exists A \in \mathcal{A})(x \in A)\}$

- $\bigcap_{A \in \mathcal{A}} A = \{x \mid (\forall A \in \mathcal{A})(x \in A)\}$

Note: The book writes $\bigcup \mathcal{A}$ for $\bigcup_{A \in \mathcal{A}} A$

and $\bigcap \mathcal{A}$ for $\bigcap_{A \in \mathcal{A}} A$.

Ex: Let $\mathcal{A} = \{\{1,2\}, \{2,3\}, \{2,5,6\}\}$. Then

$$\bigcup_{A \in \mathcal{A}} A = \{1,2\} \cup \{2,3\} \cup \{2,5,6\} = \{1,2,3,5,6\}$$

and

$$\bigcap_{A \in \mathcal{A}} A = \{1,2\} \cap \{2,3\} \cap \{2,5,6\} = \{2\}.$$

Ex: Let $A_n = \{k \in \mathbb{N} \mid k \geq n\}$
$= \{n, n+1, n+2, \ldots\}$

So $A_1 = \{1, 2, 3, \ldots\} = \mathbb{N}$
$A_2 = \{2, 3, 4, \ldots\}$
$A_3 = \{3, 4, 5 \ldots\}$
$\vdots$

Set $\mathcal{A} = \{A_n \mid n \in \mathbb{N}\}$
$= \{A_1, A_2, A_3, \ldots\}$.

A set with infinitely many elements, each of which is a set

Then

$$\bigcup_{A \in \mathcal{A}} A = \bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup A_3 \cup \cdots = \mathbb{N}.$$

Proof: Let $x \in \bigcup_{n=1}^{\infty} A_n$. Then $x \in A_n$ for some n. But $A_n \subseteq \mathbb{N}$, so $x \in \mathbb{N}$. Thus, $\bigcup_{i=1}^{\infty} A_i \subseteq \mathbb{N}$.

On the other hand, let $x \in \mathbb{N}$. Since $\mathbb{N} = A_1$, $x \in \bigcup_{n=1}^{\infty} A_n$. Thus, $\mathbb{N} \subseteq \bigcup_{n=1}^{\infty} A_n$. ∎

Also,

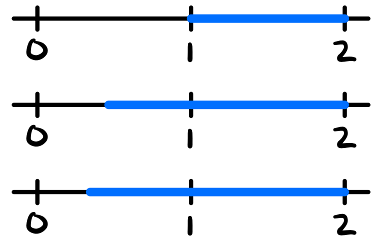
$$\bigcap_{A \in \mathcal{A}} A = \bigcap_{n=1}^{\infty} A_n = A_1 \cap A_2 \cap A_3 \cap \cdots = \emptyset.$$

Proof: Suppose $x \in \bigcap_{n=1}^{\infty} A_n$. Then $x \in A_n$ for every n. In particular, $x \in A_1 = \mathbb{N}$. But then $x \notin A_{x+1}$, which contradicts $x \in A_n$ for all $n \in \mathbb{N}$.

So $\bigcap_{n=1}^{\infty} A_n$ must be empty. ∎

Ex: Let $B_n = [\frac{1}{n}, 2]$ for each $n \in \mathbb{N}$.

$B_1 = [1,2]$
$B_2 = [\frac{1}{2}, 2]$
$B_3 = [\frac{1}{3}, 2]$
$\vdots$



Then $\bigcap_{i=1}^{\infty} B_n = [1,2]$.

Proof: Left to you.

And $\bigcup_{i=1}^{\infty} B_n = (0,2]$.

Proof: Each $B_n \subseteq (0,2]$, so $\bigcup_{n=1}^{\infty} B_n \subseteq (0,2]$
why?

Now, let $x \in (0,2]$.
By the Archimedean Property (Bonus Problem #6), there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < x$.

Thus, $x \in B_m = [\frac{1}{m}, 2]$, and so $x \in \bigcup_{n=1}^{\infty} B_n$.
That is, $(0,2] \subseteq \bigcup_{n=1}^{\infty} B_n$. ∎

Ex: Similarly, if $C_n = (-\frac{1}{n}, 2]$, then

$$\bigcup_{n=1}^{\infty} C_n = (-1, 2] \quad \text{and} \quad \bigcap_{n=1}^{\infty} C_n = [0, 2].$$

Thm: Let $\mathcal{A}$ be a non-empty set of sets. Let $A_0 \in \mathcal{A}$. Then

$$\bigcap_{A \in \mathcal{A}} A \underset{①}{\subseteq} A_0 \underset{②}{\subseteq} \bigcup_{A \in \mathcal{A}} A.$$

Proof: ① Let $x \in \bigcap_{A \in \mathcal{A}} A$. Then for all $A \in \mathcal{A}$, $x \in A$. In particular, $x \in A_0$. Thus, $\bigcap_{A \in \mathcal{A}} A \subseteq A_0$.

② Let $x \in A_0$. Then there exists some $A \in \mathcal{A}$ such that $x \in A$, because we could take $A = A_0$. This means $x \in \bigcup_{A \in \mathcal{A}} A$. Therefore, $A_0 \subseteq \bigcup_{A \in \mathcal{A}} A$. ∎

Thm (Generalized DeMorgan Laws for sets):
Let S be a set and let $\mathcal{A}$ be a set of sets.
Then

(i) $S \setminus \left(\bigcup_{A \in \mathcal{A}} A \right) = \bigcap_{A \in \mathcal{A}} (S \setminus A)$

(ii) $S \setminus \left(\bigcap_{A \in \mathcal{A}} A \right) = \bigcup_{A \in \mathcal{A}} (S \setminus A)$.

Thm (Generalized Distributive Laws for sets):
Let S be a set and let $\mathcal{A}$ be a set of sets.
Then

(i) $S \cap \left(\bigcup_{A \in \mathcal{A}} A \right) = \bigcup_{A \in \mathcal{A}} (S \cap A)$

(ii) $S \cup \left(\bigcap_{A \in \mathcal{A}} A \right) = \bigcap_{A \in \mathcal{A}} (S \cup A)$.

The Power Set

<u>Def</u>: Let A be a set. The <u>power set</u> of A, denoted $\mathcal{P}(A)$ is the set of all subsets of A.

$$\mathcal{P}(A) = \{S \mid S \subseteq A\}.$$

<u>Ex</u>: $A = \{1,2\}$. Then $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$.

If A has n elements, then $\mathcal{P}(A)$ has 2^n elements.

Ordered Pairs

"Def": An <u>ordered pair</u> is a list of two objects in order.
If a and b are objects, then (a,b) denotes the ordered pair with first entry a and second entry b.

What do we mean by "in order"?

<u>Fundamental Property</u>: $(a,b)=(c,d)$ if and only if $a=c$ and $b=d$.

<u>Ex</u>: • If $a \neq b$, then $(a,b) \neq (b,a)$.
• For any a, (a,a) is a perfectly fine ordered pair.

Compare with sets:

- $\{a,b\} = \{b,a\}$
- $\{a,a\} = \{a\}$

<u>Aside</u>: There is an "implementation" of ordered pairs as sets. To do this, <u>define</u>

$$(a, b) = \{\{a\}, \{a,b\}\}.$$

Then you can <u>prove</u> that $(a,b) = (c,d) \iff a=c$ and $b=d$.