Cartesian Products

Def: Let A and B be sets. The <u>Cartesian product</u> of A and B is the set

$$A \times B = \{(a,b) \mid a \in A,\ b \in B\}.$$

Ex: Let $A = \{a,b,c\}$ and $B = \{2,4\}$. Then

$$A \times B = \{(a,2),\ (a,4),\ (b,2),\ (b,4),\ (c,2),\ (c,4)\}.$$

We write $A^2 = A \times A$.

Ex: $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$

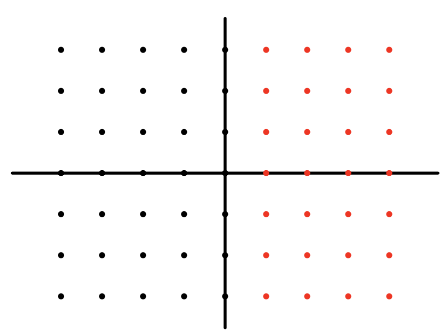
is the usual <u>Cartesian plane</u>.

Ex:

$$\mathbb{N} \times \mathbb{Z} = \{(m,n) \mid m \in \mathbb{N},\ n \in \mathbb{Z}\}.$$

$$\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z} = \{(m,n) \mid m \in \mathbb{Z},\ n \in \mathbb{Z}\}.$$

<u>Picture</u>:



Note that $\mathbb{N} \times \mathbb{Z} \subseteq \mathbb{Z}^2 \subseteq \mathbb{R}^2$.

For sets A, B, C, we can similarly define

$$A \times B \times C = \{ (a,b,c) \mid a \in A,\ b \in B,\ c \in C \}.$$

ordered triples

More generally, we can define the Cartesian product of n sets to be the set of ordered <u>n-tuples</u>.

<u>Ex</u>: $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{ (x,y,z) \mid x, y, z \in \mathbb{R} \}.$

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n} = \{ (x_1, x_2, \dots, x_n) \mid \text{each } x_i \in \mathbb{R} \}.$$

<u>Functions</u>

<u>Def</u>: Let A and B be sets. A <u>function</u> $f: A \to B$ is a rule which associates to each $x \in A$ an element $f(x) \in B$.

- A is the <u>domain</u> of f, written $A = \mathrm{Dom}(f)$.
  (the set of all valid inputs)
  We might say f is a <u>function on A</u>.

- B is the <u>target</u> or <u>codomain</u> of f.
  (a set containing all possible outputs)

- For $x \in A$, $f(x)$ is the <u>value</u> of f at x.
  [f is the <u>function</u>, $f(x)$ is an <u>element</u> of B]

- The word <u>map</u> is a synonym for function.

Note that to define a function, we must specify both the domain and the target.

Def: Let $f: A \to B$ be a function. The range of f, denoted $\mathrm{Rng}(f)$, is the set

$$\mathrm{Rng}(f) = \{y \in B \mid f(x) = y \text{ for some } x \in A\}.$$

Note: $\mathrm{Rng}(f) \subseteq B$ automatically.

Informally, $\mathrm{Rng}(f)$ is the set of all function values.

Ex: Let $A = \{a, b, c, d\}$. Define $f: A \to \mathbb{Z}$ by

$$f(a) = 2, \quad f(b) = 3, \quad f(c) = 1, \quad f(d) = 1.$$

When the domain is finite, like it is here, we can represent the function as a table.

Dummy variable

x	$f(x)$
a	2
b	3
c	1
d	1

$$\mathrm{Rng}(f) = \{1, 2, 3\} \subseteq \mathbb{Z}.$$

Ex: Consider the functions

Dummy variable

$f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$

$g: \mathbb{R} \to [0,\infty)$ " " $g(x) = x^2$

$h: \mathbb{R} \to [-2,\infty)$ " " $h(x) = x^2$

$i: [0,\infty) \to [0,\infty)$ " " $i(x) = x^2$

$j: [1,2] \to \mathbb{R}$ " " $j(x) = x^2$

$k: [1,2] \to [1,4]$ " " $k(x) = x^2$

Q: Why can't we add $\ell: [1,3] \to [1,5]$ given by $\ell(x) = x^2$ to this list?

Def: We say two functions f and g are <u>equal</u> if

① $\mathrm{Dom}(f) = \mathrm{Dom}(g)$

and

② For every $x \in \mathrm{Dom}(f)$, $\underline{f(x) = g(x)}$.

equality of values

In this case we write $\underline{f = g}$.

equality of functions

Ex: In the previous example, we have 3 functions up to equality: $f = g = h$, i, and $j = k$.

Ex: Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$.
Then $\operatorname{Rng}(f) = [0, \infty)$.

Proof: ($\subseteq$) Let $y \in \operatorname{Rng}(f)$. Then $y \in \mathbb{R}$ and $y = f(x)$ for some $x \in \mathbb{R}$. Thus $y = x^2 \geq 0$, so $y \in [0, \infty)$.

($\supseteq$) Let $y \in [0, \infty)$. Then $y \geq 0$, so $\sqrt{y} \in \mathbb{R}$. Set $x \in \sqrt{y}$. We have

$$f(x) = x^2 = (\sqrt{y})^2 = y,$$

which shows that $y \in [0, \infty)$.

$\blacksquare$