Warm-Up: Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ be given by

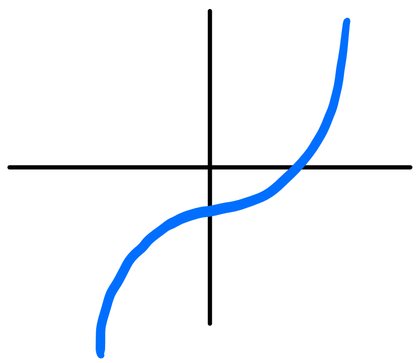
$$f(m,n) = m - n.$$

e.g. $f(1,3) = -2$, $f(3,1) = 2$.

Show that $\mathrm{Rng}(f) = \mathbb{Z}$.

Graphs

Ex: For $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 - 2$, the graph of f is



What is this? It's

$$\{(x,y) \in \mathbb{R} \times \mathbb{R} \mid y = x^3 - 2\}.$$

Def: Let $f: A \to B$ be a function.
The *graph* of f is

$$\mathrm{Graph}(f) = \{(x,y) \in A \times B \mid y = f(x)\}.$$

Observe: For each $x \in A$, there is a *unique* $y \in B$ such that $(x,y) \in \mathrm{Graph}(f)$, namely $y = f(x)$.
"vertical line test"

Aside: You can actually use graphs to *define* what a function is.

Def: Let A and B be sets. A *function* $f: A \to B$ is a subset

$$\mathrm{Graph}(f) \subseteq A \times B$$

with the property that for all $x \in A$, there exists a unique $y \in B$ such that $(x,y) \in \mathrm{Graph}(f)$.

If $(x,y) \in \mathrm{Graph}(f)$, write $f(x) = y$.

Note: • You don't have to use this definition. But it's more concrete than defining a function as a "rule."

• We can't always draw Graph(f).

• $\mathrm{Rng}(f) = \{ y \in B \mid (x,y) \in \mathrm{Graph}(f) \}$.

Ex: Let S be any set. Define a function

$$id_S : S \to S$$

by $id_S(x) = x$ for all $x \in S$.

This is called the __identity function__ on S.

Function Composition

Def: Let $f : A \to B$ and $g : B \to C$ be functions. The composition of g with f is the function

$$g \circ f : A \to C$$

given by

$$(g \circ f)(a) = g(f(a))$$

"g after f"

for all $a \in A$.

Ex: $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2^x$
$g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^2$.

Then $(g \circ f)(x) = (2^x)^2 = 2^{2x} = 4^x$

and

$$(f \circ g)(x) = 2^{(x^2)}$$

$g \circ f \neq f \circ g$, since $(g \circ f)(1) = 4$
$(f \circ g)(1) = 2$

Order matters!

<u>Note</u>: Read compositions from right to left

- Sometimes, $g \circ f$ is defined but $f \circ g$ is not

<u>Picture</u>:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

<u>Thm</u>: Let $f: A \to B$, $g: B \to C$, and $h: C \to D$ be functions. Then

$$(h \circ g) \circ f = h \circ (g \circ f)$$

<u>Proof idea</u>: Both are given by $x \mapsto h(g(f(x)))$.

Ex: If $f: A \to S$ and $g: S \to B$,
then

$$\mathrm{id}_S \circ f = f$$

and

$$g \circ \mathrm{id}_S = g.$$