Surjections

Def: Let $f: A \to B$ be a function. We say f is a <u>surjection</u> if for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

Also say: f is <u>surjective</u>, f is <u>onto</u>.

Equivalently: $f: A \to B$ is surjective $\iff$ $Rng(f) = B$

Ex: Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$. Then f is a surjection.

Proof: Let $y \in \mathbb{R}$. Set $x = \sqrt[3]{y} \in \mathbb{R}$. Then

$$f(\sqrt[3]{y}) = (\sqrt[3]{y})^3 = y. \qquad \blacksquare$$

Ex: Let $g: \mathbb{R} \to \mathbb{R}$ be given by $g(x) = x^2$. Then g is not surjective.

Proof: Consider $-1 \in \mathbb{R}$. Then for all $x \in \mathbb{R}$, $g(x) = x^2 \neq -1$.

Note: f surjective $\iff (\forall y \in B)(\exists x \in A)(f(x) = y)$

f not surjective $\iff (\exists y \in B)(\forall x \in A)(f(x) \neq y)$.

Ex: However, $h : \mathbb{R} \to [0, \infty)$ given by $h(x) = x^2$ is surjective.

Injections

Def: A function $f : A \to B$ is an injection if for all $x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Also say: f is injective, f is one-to-one.

Ex: $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is an injection.

Proof: Let $x_1, x_2 \in \mathbb{R}$ and suppose $x_1^3 = x_2^3$. Then $\sqrt[3]{x_1^3} = \sqrt[3]{x_2^3}$, i.e. $x_1 = x_2$. $\blacksquare$

Ex: $g: \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^2$ is **not** an injection.

Proof: Consider $-1, 1 \in \mathbb{R}$. Then $-1 \neq 1$, but
$g(-1) = (-1)^2 = 1 = (1)^2 = g(1)$. ∎

Note: f injective $\iff (\forall x_1, x_2 \in A)[\underset{T}{f(x_1) = f(x_2)} \Rightarrow \underset{F}{x_1 = x_2}]$

$\iff (\forall x_1, x_2 \in A)[\underset{T}{x_1 \neq x_2} \Rightarrow \underset{F}{f(x_1) \neq f(x_2)}]$

f not injective $\iff (\exists x_1, x_2 \in A)[x_1 \neq x_2 \text{ and } f(x_1) = f(x_2)]$

Ex: However, $h: [0, \infty) \to \mathbb{R}$ given by $h(x) = x^2$ **is** injective.

Bijections

Def: A function $f: A \to B$ is a bijection if it is both a surjection and an injection.

f surjective $\iff (\forall y \in B)(\exists x \in A)[f(x) = y]$

f injective $\iff (\forall x_1, x_2 \in A)[f(x_1) = f(x_2) \Rightarrow x_1 = x_2]$

$\iff (\forall y \in B)(\forall x_1, x_2 \in A)[(f(x_1) = y \wedge f(x_2) = y) \Rightarrow x_1 = x_2]$

Together, we get the following:

Lemma: Let $f: A \to B$ be a function. Then f is a bijection if and only if for every $y \in B$, there exists a unique $x \in A$ such that $f(x) = y$.

Ex: $f: \mathbb{Z} \to \mathbb{Z}$. For each $y \in \mathbb{Z}$, there is a unique $x \in \mathbb{Z}$ such that $f(x) = y$, namely $x = y - 1$.

$n \longmapsto n+1$