Warm-Up: Prove that

$$f : \mathbb{R}\setminus\{3\} \to \mathbb{R}\setminus\{1\}$$
$$x \mapsto \frac{x}{x-3}$$

is a bijection.

Inverse Functions

A bijection $f: A \to B$ gives us a rule for going back to B from A. Specifically, $y \in B$ can map back to the unique $x \in A$ such that $f(x) = y$.

Def: Let $f: A \to B$ be a bijection. The inverse function of f is

$$f^{-1} : B \to A$$

defined as follows: For each $y \in B$, $f^{-1}(y)$ is the unique element $x \in A$ such that $f(x) = y$.

That is $f^{-1}(y) = x \iff y = f(x)$.

Ex: $f: \mathbb{R} \to (0,\infty)$ given by $f(x) = e^x$ is a bijection.

$f^{-1}: (0,\infty) \to \mathbb{R}$ is given by $f^{-1}(y) = \ln(y)$.

$\ln(y) = x \iff y = e^x$

Ex: $g: [0,\infty) \to [0,\infty)$ is a bijection.
$x \longmapsto x^2$

Its inverse is $g^{-1}: [0,\infty) \to [0,\infty)$
$y \longmapsto \sqrt{y}$

$\sqrt{y} = x \iff y = x^2$ and $x \geq 0$

Ex: $\sin: \mathbb{R} \to \mathbb{R}$ is not a bijection, but $\sin: [-\frac{\pi}{2}, \frac{\pi}{2}] \to [1,1]$ is.

Its inverse is $\sin^{-1}: [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\sin^{-1}(y) = x \iff y = \sin(x)$ and $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

Thm: Let $f: A \to B$ be a bijection and let $f^{-1}: B \to A$ be its inverse. Then

$$① \quad f^{-1} \circ f = id_A : A \to A$$

and

$$② \quad f \circ f^{-1} = id_B : B \to B$$

This is essentially a rephrasing of the fundamental identity $f^{-1}(y) = x \iff f(x) = y$.

Proof: ① Let $x \in A$. We must show

$$(f^{-1} \circ f)(x) = id_A(x) = x.$$

Set $y = f(x)$. Then, by definition of f^{-1}, $f^{-1}(y) = x$. But then

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x. \quad ✓$$

② Let $y \in B$ We must show

$$(f \circ f^{-1})(y) = id_B(y) = y.$$

Set $x = f^{-1}(y)$. Then $f(x) = y$, so

$$(f \circ f^{-1})(y) = f(f^{-1}(y)) = f(x) = y. \quad ✓$$

∎

Cor: Let $f: A \to B$ be a bijection. Then its inverse $f^{-1}: B \to A$ is also a bijection, and $(f^{-1})^{-1} = f$.

Proof: Let $f: A \to B$ be a bijection.

- **f^{-1} is surjective**: Let $x \in A$.

  We must find $y \in B$ so that $f^{-1}(y) = x$.

  Set $y = f(x)$. Then, by the theorem,

  $$f^{-1}(y) = f^{-1}(f(x)) = x. \quad \checkmark$$

- **f^{-1} is injective**: Let $y_1, y_2 \in B$ such that $f^{-1}(y_1) = f^{-1}(y_2)$.

  Then

  $$f(f^{-1}(y_1)) = f(f^{-1}(y_2)),$$

  so by the theorem,

  $$y_1 = y_2. \quad \checkmark$$

- **$(f^{-1})^{-1} = f$**: By definition, for $x \in A$ and $y \in B$,

  $$(f^{-1})^{-1}(x) = y \iff x = f^{-1}(y) \iff f(x) = y.$$

  Thus, $(f^{-1})^{-1} = f$. ■

The following theorems are proved using similar methods.

Thm: Let $f: A \to B$ and $g: B \to A$ be functions. If
$$g \circ f = id_A \quad \text{and} \quad f \circ g = id_B,$$
then f is a bijection and $g = f^{-1}$.

Thm: If $f: A \to B$ and $g: B \to C$ are bijections, then $g \circ f: A \to C$ is a bijection also, and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.