Warm-Up: Prove that the function

$$f: \mathbb{N} \longrightarrow \mathbb{N}\setminus\{1\}$$
$$x \longmapsto x+1$$

is a bijection.

Cardinality

What does it mean for a set A to have exactly n elements?

Ex: $A = \{4, \text{red}, \$\}$ has exactly 3 elements

How do we know? We can list them:

1. 4
2. red
3. \$

This is just a bijection $f: \{1,2,3\} \to A$.

surjection $\Leftrightarrow$ every element in A is on the list

injection $\Leftrightarrow$ no element in A is on the list more than once.

Def: Let A and B be sets. We say A and B <u>have the same cardinality</u>, denoted $|A| = |B|$, if there exists a bijection $f: A \to B$.

<u>Book</u>: A and B are <u>equinumerous</u>, $\bar{\bar{A}} = \bar{\bar{B}}$.

If A is a set and $n \in \mathbb{N}$ such that A and $\{1, 2, \dots, n\}$ have the same cardinality, then we say A has <u>cardinality n</u> (or A has <u>exactly n elements</u>), and write $|A| = n$.

We also write $|\emptyset| = 0$.

This is an equivalence relation.

Thm: Let A, B, C be sets. Then

1. $|A| = |A|$. [Reflexive]
2. If $|A| = |B|$, then $|B| = |A|$. [Symmetric]
3. If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$. [Transitive]

Proof sketch: ① $\mathrm{id}_A : A \to A$, $x \mapsto x$ is a bijection.

② If $f: A \to B$ is a bijection, then $f^{-1}: B \to A$ is a bijection.

③ If $f: A \to B$ and $g: B \to C$ are bijections, then $g \circ f : A \to C$ is a bijection. (HW 22)

Finite and infinite sets

Def: A set A is *finite* if either
- $A = \emptyset$ (i.e. $|A| = 0$)

or
- there exists $n \in \mathbb{N}$ such that $|A| = n$.

A set is *infinite* if it is not finite.

Ex: $A = \{4, \text{red}, \$\}$. $|A| = 3$

Ex: $B = \{a, b, c, \ldots, z\}$. $|B| = 26$

Ex: Is there a set C such that $|C| = 3$ and $|C| = 26$? No!

Thm: Let $n, m \in \mathbb{N} \cup \{0\}$. If A is a set such that $|A| = n$ and $|A| = m$, then $n = m$.

Proof idea:

$|A| = n$ means there is a bijection $f: \{1, \dots, n\} \to A$

$|A| = m$ " " " $g: \{1, \dots, m\} \to A$.

So $g^{-1} \circ f: \{1, \dots, n\} \to \{1, \dots, m\}$ is a bijection.

- If $n > m$, this can't be injective.
- If $n < m$, this can't be surjective.