Warm-Up: Can you find a bijection

$$f : \mathbb{N} \to \mathbb{Z} ?$$

Infinite Sets

We already saw that

$$f : \mathbb{N} \to \mathbb{N} \setminus \{1\}$$
$$x \mapsto x+1$$

is a bijection, so $|\mathbb{N}| = |\mathbb{N} \setminus \{1\}|$.

Here's another example:

Ex: Let $E = \{ n \in \mathbb{N} \mid n \text{ is even} \} = \{2, 4, 6, 8, \ldots\}$.
Then

$$g : \mathbb{N} \to E$$
$$x \mapsto 2x$$

is a bijection. Thus, $|\mathbb{N}| = |E|$.

Proof: Let $x_1, x_2 \in \mathbb{N}$. If $f(x_1) = f(x_2)$, then $2x_1 = 2x_2$, so cancelling the 2 gives $x_1 = x_2$. Thus, f is injective.

Let $y \in E$. Then $y = 2k$ for some $k \in \mathbb{N}$ (why?) Thus, $f(k) = 2k = y$. This shows that f is surjective. ■

So for infinite sets, it's possible to find a bijection $f: A \to B$ (i.e. $|A| = |B|$) when $A \subsetneq B$ or $B \subsetneq A$.

"Hilbert's Hotel"

Def: A set A is <u>countably infinite</u> if there exists a bijection $f: \mathbb{N} \to A$. That is, if $|A| = |\mathbb{N}|$.

A set is <u>countable</u> if it is finite or countably infinite.

A set is <u>uncountable</u> if it is not countable.

Ex: $\mathbb{N}$ is countably infinite

$\mathbb{N} \setminus \{1\}$ is countably infinite

$E = \{n \in \mathbb{N} \mid n \text{ is even}\}$ is countably infinite.

$\mathbb{Z}$ is countably infinite

Think: Countably infinite sets can be enumerated in an infinite list.

Uncountable sets (if they exist) cannot be listed, even if the list is infinite!

Thm: Let A be a countably infinite set. Then any subset $B \subseteq A$ is countable.

Ex: $\mathbb{N} \times \mathbb{N}$ is countably infinite

Key: Write the elements of $\mathbb{N} \times \mathbb{N}$ in a grid

	1	2	3	4	5	$\cdots$
1	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	
2	$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$	$(2,5)$	
3	$(3,1)$	$(3,2)$	$(3,3)$	$(3,4)$	$(3,5)$	
4	$(4,1)$	$(4,2)$	$(4,3)$	$(4,4)$	$(4,5)$	
5	$(5,1)$	$(5,2)$	$(5,3)$	$(5,4)$	$(5,5)$	
$\vdots$						$\ddots$

Define a bijection $f: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by reading along the northeast diagonals in order:

$$f(1) = (1,1)$$
$$f(2) = (2,1)$$
$$f(3) = (1,2)$$
$$f(4) = (3,1)$$
$$\vdots$$

Ex: The set $\mathbb{Q}_{>0} = \{ q \in \mathbb{Q} \mid q > 0 \}$ of positive rational numbers is countably infinite.

__Key idea__: Each $q \in \mathbb{Q}_{>0}$ can be written __uniquely__ as $q = \frac{a}{b}$ where

- $a, b \in \mathbb{N}$

and

- $\frac{a}{b}$ is in lowest terms $(\gcd(a,b) = 1)$

Now, use a grid again, but cross out fractions not in lowest terms:

	1	2	3	4	5	$\cdots$
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	
2	$\frac{2}{1}$	$\cancel{\frac{2}{2}}$	$\frac{2}{3}$	$\cancel{\frac{2}{4}}$	$\frac{2}{5}$	
3	$\frac{3}{1}$	$\frac{3}{2}$	$\cancel{\frac{3}{3}}$	$\frac{3}{4}$	$\frac{3}{5}$	
4	$\frac{4}{1}$	$\cancel{\frac{4}{2}}$	$\frac{4}{3}$	$\cancel{\frac{4}{4}}$	$\frac{4}{5}$	
5	$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\cancel{\frac{5}{5}}$	
$\vdots$						$\ddots$

Define a bijection $g : \mathbb{N} \to \mathbb{Q}_{>0}$ by reading the remaining entries along the northeast diagonals:

$$g(1) = 1, \quad g(2) = 2, \quad g(3) = \tfrac{1}{2}, \quad g(4) = 3, \quad g(5) = \tfrac{1}{3}, \quad \ldots$$

Ex: $\mathbb{Q}$ is countably infinite.

Let $g: \mathbb{N} \to \mathbb{Q}_{>0}$ be the bijection above.
Define a bijection $h: \mathbb{N} \to \mathbb{Q}$ by

$$h(n) = \begin{cases} 0 & \text{if } n = 1 \\ g\left(\frac{n}{2}\right) & \text{if } n \text{ is even} \\ -g\left(\frac{n-1}{2}\right) & \text{if } n \text{ is odd} \end{cases}$$

So

$$\begin{aligned} h(1) &= 0 \\ h(2) &= g(1) = 1 \\ h(3) &= -g(1) = -1 \\ h(4) &= g(2) = 2 \\ h(5) &= -g(2) = -2 \\ &\vdots \end{aligned}$$