Uncountability of $\mathbb{R}$

Thm: $|\mathbb{R}| \neq |\mathbb{N}|$ ($\mathbb{R}$ is uncountable)

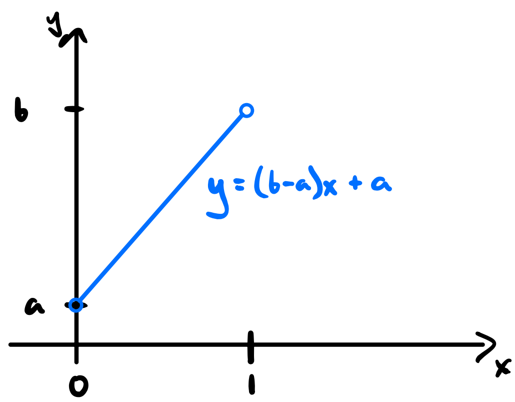
Step 1: If $a, b \in \mathbb{R}$ with $a < b$, then $|(a,b)| = |(0,1)|$.

We must give a bijection between $(0,1)$ and (a,b).

A linear function will work:

$$f : (0,1) \to (a,b)$$
$$x \mapsto (b-a)x + a$$

Graph:



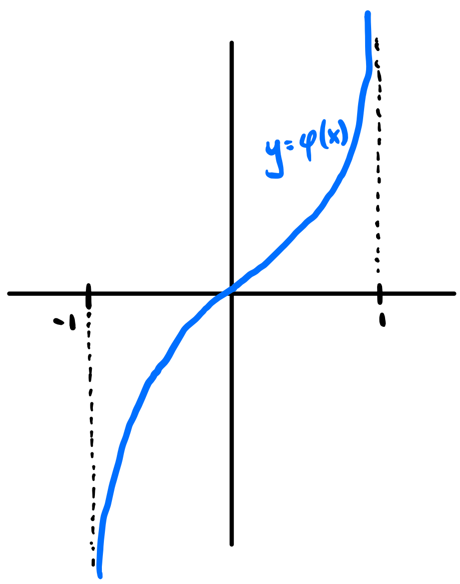
Exercise: Check that f is a bijection.

Step 2: $|\mathbb{R}| = |(-1,1)|$

There are many ways to do this, but we'll use the book's: Define

$$\varphi : (-1,1) \to \mathbb{R}$$
$$x \longmapsto \frac{x}{1-|x|}$$

Graph:



Exercise: Check that φ is a bijection.

(See HW 23 practice problem 2. Basically follows from HW 23 exercise 1.)

Step 3: There is no surjection $\mathbb{N} \to (0,1)$ (and thus no bijection $\mathbb{N} \to (0,1)$).

Why is this enough? If $|\mathbb{N}| = |\mathbb{R}|$, then since $|\mathbb{R}| = |(-1,1)|$ and $|(-1,1)| = |(0,1)|$, transitivity gives $|\mathbb{N}| = |(0,1)|$, a contradiction.

To show this, we use Cantor's Diagonal Argument.

Need:
- Every real number has an infinite decimal representation.

$$\text{e.g.} \quad \frac{1}{3} = 0.3333333\cdots$$
$$\frac{3}{4} = 0.7500000\cdots$$
$$\pi - 3 = 0.14159265\cdots$$

- This representation is unique if we don't allow infinite repeating 9s.

$$\text{e.g.} \quad \frac{3}{4} = 0.749999999\cdots$$
$$= 0.75000000\cdots$$

Now, let $f : \mathbb{N} \to (0,1)$ be a function. Think of this as an infinite list:

$$c_1 = f(1) = 0.x_{11}x_{12}x_{13}x_{14}x_{15}\cdots$$
$$c_2 = f(2) = 0.x_{21}x_{22}x_{23}x_{24}x_{25}\cdots$$
$$c_3 = f(3) = 0.x_{31}x_{32}x_{33}x_{34}x_{35}\cdots$$
$$c_4 = f(4) = 0.x_{41}x_{42}x_{43}x_{44}x_{45}\cdots$$
$$\vdots$$

x_{nm} = mth digit of nth number

Define a number c_0 by

$$c_0 = 0.x_{01}x_{02}x_{03}x_{04}x_{05}\cdots$$

where

$$x_{0m} = \begin{cases} 1 & \text{if } x_{mm} \neq 1 \\ 2 & \text{if } x_{mm} = 1 \end{cases}$$

↳ or use any other pair of digits not including 9.

Then $c_0 \in (0,1)$, but

$$\begin{aligned} c_0 &\neq c_1 && \text{because} && x_{01} \neq x_{11} \\ c_0 &\neq c_2 && \text{"} && x_{02} \neq x_{22} \\ c_0 &\neq c_3 && \text{"} && x_{03} \neq x_{33} \\ &\vdots && && \vdots \end{aligned}$$

Thus, $c_0 \notin \mathrm{Rng}(f)$, so f is not surjective.

If time: Cantor's Generalized Diagonal Lemma

A similar argument shows that we can always find "larger" infinities.

Def: Let A and B be sets.

- We write $|A| \leq |B|$ if there exists an injection $A \to B$.

- We write $|A| < |B|$ if $|A| \leq |B|$ and $|A| \neq |B|$.

Note: This is consistent with what we know about finite sets, where $|A|$ and $|B|$ are non-negative integers.

Ex: $|\mathbb{N}| < |\mathbb{R}|$.

Thm (Cantor): Let A be any set. Then

$$|A| < |\mathcal{P}(A)|$$

Note: We've seen that if A is finite, then $|\mathcal{P}(A)| = 2^{|A|} > |A|$.

So the interesting (hard) part of this theorem is the case where A is infinite.

Proof: First, consider

$$g: A \longrightarrow \mathcal{P}(A)$$
$$x \longmapsto \{x\}$$

This is an injection, since $\{x_1\} = \{x_2\}$ if and only if $x_1 = x_2$.

Thus $|A| \leq |\mathcal{P}(A)|$.

Next, we must show $|A| \neq |\mathcal{P}(A)|$.

Consider any function $f: A \to \mathcal{P}(A)$. So for any $x \in A$, we get a subset $f(x) \subseteq A$.

Claim: f is not surjective
(Thus, f cannot be a bijection.)

Consider

$$S = \{x \in A \mid x \notin f(x)\} \subseteq A.$$

Suppose that $S \in \mathrm{Rng}(f)$. Then $S = f(x_0)$ for some $x_0 \in A$.

Is $x_0 \in S$ or $x_0 \notin S$?

- If $x_0 \in S$, then by definition $x_0 \notin f(x_0) = S$, a contradiction.

- If $x_0 \notin S = f(x_0)$, then by definition of S, $x_0 \in S$, a contradiction.

Since both possibilities lead to a contradiction, it must be that $S \notin \mathrm{Rng}(f)$. Thus, f is not surjective.

∎

As a result, we get an increasing chain

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \cdots$$

of ever larger infinities.