Warm-Up: Show that the sentence

$$(P \land Q) \Rightarrow P$$

is a tautology – that is, it is true regardless of the sentences P and Q.

Do this ① Using a truth table.
② By writing a conditional proof.

Recall: The truth value of the sentence

"If $xy > 0$, then $x > 0$ and $y > 0$"

depends on the numerical values of x and y.

We said it is false because it can be false (e.g. $x = -1$, $y = -1$), but it can also be true (e.g. $x = 1$, $y = 1$ or $x = -1$, $y = 1$).

Quantifiers let us discuss such situations.

Quantifiers

The **universal quantifier** is $\forall$, which means "for all."

If $P(x)$ is a sentence involving the variable x, then $(\forall x)\, P(x)$ is the sentence

"for all x, $P(x)$"

also read as

"for every x, $P(x)$"

"for each x, $P(x)$"

"for any x, $P(x)$."

The <u>existential quantifier</u> is $\exists$, which means "there exists."

$(\exists x)\, P(x)$ is the sentence

"there exists x such that $P(x)$"

also read as

"for some x, $P(x)$"

"for at least one x, $P(x)$."

<u>Note</u>:
- A quantifier $(\forall, \exists)$ is always "attached" to a variable, called the <u>bound variable</u>.
- A quantifier is always followed by a sentence involving the bound variable.

Ex: Let $P(x)$ be the sentence

$$\text{“}(x > 1) \Rightarrow (x^2 > 1)\text{”}$$

and let $Q(x)$ be the converse

$$\text{“}(x^2 > 1) \Rightarrow (x > 1).\text{”}$$

Then
- $(\forall x)\, P(x)$ is true.
- $(\exists x)\, P(x)$ is true.
- $(\forall x)\, Q(x)$ is false.
- $(\exists x)\, Q(x)$ is true.

Note: We should be more careful to specify which values the bound variable is allowed to take on.

The above statements are correct when x can be any real number.

To indicate this, we will write $(\forall x \in \mathbb{R})$ and $(\exists x \in \mathbb{R})$.

Ex: Which statements are true?

① $(\exists x \in \mathbb{R})(x+4=9)$

True: $x=5$.

② $(\forall x \in \mathbb{R})(x+4=9)$

False: Try $x=0$.

③ $(\exists x \in \mathbb{R})[(x+4=9) \wedge (x \neq 5)]$

False: $x+4=9 \Rightarrow x=9-4=5$

④ $(\exists x \in \mathbb{R})(x^2+6x+8 \ngtr 0)$

True: Try $x=0$.

⑤ $(\forall x \in \mathbb{R})(x^2+6x+8 \geq 0)$

Can guess and check, or complete the square:

$$x^2+6x+8 = x^2+6x+9-1$$
$$= (x+3)^2-1.$$

False: Try $x=-3$.

⑥ $(\forall x \in \mathbb{R})(x^2 + 6x + 10 \geq 0)$

True: $x^2 + 6x + 10 = (x+3)^2 + 1 \geq 1 > 0$ for all real numbers x.

Observe:
- A single example proves a $\exists$ statement.
- A single counterexample disproves a $\forall$ statement.
- To prove a $\forall$ statement or disprove a $\exists$ statement, we need an argument that works for all values.

Note: When we use a quantifier ($\forall$ or $\exists$), a bound variable is ranging over a universe of possibilities.

Usually, we should be explicit about this.

Common choices:

$\mathbb{Z}$ = the set of integers
$\mathbb{Q}$ = the set of rational numbers
$\mathbb{R}$ = the set of real numbers
$\mathbb{C}$ = the set of complex numbers

The universe matters!

Ex: $(\exists x)(x^2 = 2)$ Ambiguous

$(\exists x \in \mathbb{Z})(x^2 = 2)$ False, $\sqrt{2} \notin \mathbb{Z}$

$(\exists x \in \mathbb{R})(x^2 = 2)$ True, $\sqrt{2} \in \mathbb{R}$

Ex: $(\forall x \in \mathbb{R})(x^2 \geq 0)$ True

$(\forall x \in \mathbb{C})(x^2 \geq 0)$ False, $\sqrt{-1} \in \mathbb{C}$