Warm-Up: For each sentence, draw a number line and indicate all x-values making the sentence true.

(a) $(x > 2) \wedge (x^2 > 4)$

(b) $(x > 2) \vee (x^2 > 4)$

(c) $(x > 2) \Rightarrow (x^2 > 4)$

(d) $(x > 2) \Leftrightarrow (x^2 > 4)$

Free + Bound Variables

Let $P(x) =$ "$x^2 + 6x + 8 \geq 0$."

- Is $P(x)$ true? It depends on x.

We say that x is a free variable in the sentence $P(x)$.

Think: The sentence $P(x)$ is a function of x.

- Is $(\forall x \in \mathbb{R})\, P(x)$ true? No!

This sentence does NOT depend on x, because of the quantifier $\forall$.

In this case, we say x is a <u>bound</u> <u>variable</u> in the sentence $(\forall x \in \mathbb{R})\, P(x)$.

The quantifier $\exists$ can also bound variables: $(\exists x)\, P(x)$ does not depend on x.

<u>Analogy</u>: $f(x) = x^2$ vs. $\int_0^1 x^2\, dx$

<u>Note</u>: Over a finite set (universe),

- $\forall$ is an "and" statement
- $\exists$ is an "or" statement

Ex: If $A = \{-3, 1, 4\}$, then

$$(\forall x \in A)(x^2 < 20) \equiv ((-3)^2 < 20) \wedge (1^2 < 20) \wedge (4^2 < 20)$$

$$(\exists x \in A)(x > 0) \equiv (-3 > 0) \vee (1 > 0) \vee (4 > 0)$$

(Both true)

For this reason, we can think of $\forall$ as "generalized and" and $\exists$ as "generalized or."

Thm (Generalized DeMorgan's Laws)

(a) $\neg[(\forall x \in A)\ P(x)] \equiv (\exists x \in A)(\neg P(x))$

(b) $\neg[(\exists x \in A)\ P(x)] \equiv (\forall x \in A)(\neg P(x))$

Proof: (a) Suppose $\neg[(\forall x \in A)\ P(x)]$ is true.
Then $(\forall x \in A)\ P(x)$ is false.

So there is some $x_0 \in A$ such that $P(x_0)$ is false, i.e. $\neg P(x_0)$ is true.

Hence $(\exists x \in A)(\neg P(x))$ is true.

Conversely, suppose $(\exists x \in A)(\neg P(x))$ is true.

Then there is $x_0 \in A$ such that $\neg P(x_0)$ is true, i.e. $P(x_0)$ is false.

So $(\forall x \in A) P(x)$ is false. Therefore, $\neg (\forall x \in A) P(x)$ is true.

(b) is similar (see book). $\boxtimes$

Thm (Generalized Distributive Laws):
Let P be a sentence not involving x.
Let $Q(x)$ be a sentence involving x.

Then

a) $P \wedge [(\exists x \in A)\, Q(x)] \equiv (\exists x \in A)[P \wedge Q(x)]$

b) $P \vee [(\forall x \in A)\, Q(x)] \equiv (\forall x \in A)[P \vee Q(x)]$.

Proof: Omitted (see book).

Order of Quantifiers

Suppose $P(x,y)$ is a sentence involving 2 variables.
What is the difference between

(a) $(\forall x)\left[(\exists y)\ P(x,y)\right]$

and

(b) $(\exists y)\left[(\forall x)\ P(x,y)\right]$?

implicit

Ex: $P(x,y) =$ " $x+y=1$ "

(a) is "for any $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $x+y=1$" True!

Proof: Let $x \in \mathbb{R}$. Set $y = 1-x$. Then $y \in \mathbb{R}$ and $x+y = x+(1-x) = 1$. ∎

(b) is "there is $y \in \mathbb{R}$ such that for any $x \in \mathbb{R}$, we have $x+y=1$" False!

How to prove? Let's show $\neg$(b) is true.

By De Morgan,

$$\neg (\exists y)\left[(\forall x)\, P(x,y)\right] \equiv (\forall y)\, \neg\left[(\forall x)\, P(x,y)\right]$$
$$\equiv (\forall y)\left[(\exists x)\, \neg P(x,y)\right]$$

Proof: Let $y \in \mathbb{R}$. We must show there is $x \in \mathbb{R}$ such that $x+y \neq 1$. Take $x = -y$. Then $x+y = (-y)+y = 0 \neq 1$.

To summarize:

$$\boxed{\begin{array}{ll} \text{(a)} & (\forall x)\left[(\exists y)\, P(x,y)\right] \\ \text{and} & \\ \text{(b)} & (\exists y)\left[(\forall x)\, P(x,y)\right] \end{array}}$$

In (a), we choose y after we know x.

In (b), we choose y first, and it has to work with every x.