Warm-up: What is the difference between

(a) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x \leq y)$

(b) $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(x \leq y)$?

Is either true?

Thm: Let $P(x,y)$ be a sentence depending on $x \in A$ and $y \in B$. Then

$$(\exists y \in B)(\forall x \in A)\, P(x,y) \Rightarrow (\forall x \in A)(\exists y \in B)\, P(x,y).$$

Proof: Assume $(\exists y \in B)(\forall x \in A)\, P(x,y)$ is true.

Then there is some $y_0 \in B$ such that

$$(\forall x \in A)\, P(x, y_0) \text{ is true.}$$

That is, for each $x \in A$, $P(x, y_0)$ is true.

Then $(\exists y \in B) P(x,y)$ is true for each $x \in A$, because we can take $y = y_0$.

In other words, $(\forall x \in A)(\exists y \in B) P(x,y)$ is true. ∎

Ex: $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(xy = 0)$

True: Take $y = 0$.

Thus, $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(xy = 0)$ is also true.

Quantifiers of the same type commute:

Thm: Let $P(x,y)$ be a sentence depending on $x \in A$ and $y \in B$. Then

① $(\forall x \in A)(\forall y \in B)\, P(x,y) \equiv (\forall y \in B)(\forall x \in A)\, P(x,y)$

② $(\exists x \in A)(\exists y \in B)\, P(x,y) \equiv (\exists y \in B)(\exists x \in B)\, P(x,y)$

Proof: Talk it out.

Note: When $B = A$, often write $(\forall x, y \in A)\, P(x,y)$ instead of $(\forall x \in A)(\forall y \in A)\, P(x,y)$.

Unique Existence

The **unique existential quantifier** is $\exists!$:

$(\exists! x \in A)\, P(x)$ means

"There exists a unique (i.e. one and only one) $x \in A$ such that $P(x)$."

Note: $\exists!$ is "generalized exclusive or"

Ex: ① $(\exists! x \in \mathbb{R})(x^2 = 0)$

True: $x^2 = 0 \iff x = 0$.

② $(\exists! x \in \mathbb{R})(x^2 = 2)$

False: $x = \sqrt{2}$ and $x = -\sqrt{2}$ each satisfy $x^2 = 2$.

Uniqueness fails.

③ $(\exists! x \in \mathbb{R})(x^2 = -2)$

False: $x^2 = -2$ has no solutions in $\mathbb{R}$.

Existence fails.

④ $(\forall x \in \mathbb{R})\left[x \neq 0 \Rightarrow (\exists! y \in \mathbb{R})(xy = 1)\right]$

True: If $x \neq 0$, then

$xy = 1 \Leftrightarrow y = \frac{1}{x}$.

Observation: $\exists!$ can be written in terms of $\forall$ and $\exists$:

$$(\exists!\, x \in A)\, P(x) \equiv (\exists x \in A)\left[P(x) \wedge \underbrace{(\forall y \in A)(P(y) \Rightarrow (x = y))}_{\text{Any other solution is the one we already have } (x).}\right]$$

Induction

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of <u>natural numbers</u>.

<u>Ex</u>: You might have seen the following formula in Calc II:

For each $n \in \mathbb{N}$,

$$\underbrace{1 + 2 + 3 + \cdots + n}_{= \sum_{i=1}^{n} i} = \frac{n(n+1)}{2}$$

How can we prove $(\forall n \in \mathbb{N})\left(\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\right)$?

We need an argument that works for <u>every</u> n — but as n gets larger we get more and more summands.

<u>$n=1$</u>: $1 = \frac{1 \cdot 2}{2}$

<u>$n=2$</u>: $1+2 = 3 = \frac{2 \cdot 3}{2}$

<u>$n=3$</u>: $1+2+3 = 6 = \frac{3 \cdot 4}{2}$

etc.