1 (Judson 9.5 – modified)

(a) Prove that $U(5) \cong U(10)$ by explicitly defining an isomomorphism

$$\varphi\colon U(5) \to U(10).$$

Be sure to check that φ is an isomorphism!

(b) Prove that $U(5) \ncong U(12)$.

2 Let G be a group and let $g \in G$ be an element. Define the "left multiplication by g" map

$$\begin{aligned} \lambda_g \colon G &\to G \\ x &\to gx. \end{aligned}$$

(a) Prove that λ_g is a bijection from G to itself.

(b) Prove that λ_g is an isomorphism if and only if $g = e$ is the identity element in G.

3 Let G be a group and let $g \in G$ be an element. Define the "conjugation by g" map

$$\begin{aligned}\gamma_g \colon G &\to G\\ x &\mapsto gxg^{-1}.\end{aligned}$$

(a) Recall that an isomorphism from G to itself is called an **automorphism** of G. Prove that γ_g is an automorphism.

(b) Prove that

$$\operatorname{Inn}(G) = \{\gamma_g \mid g \in G\}$$

is a group under function composition. This group is called the **inner automorphism group** of G.