1 Let $S = \{f\colon \mathbb{R} \to \mathbb{R}\}$ be the set of all functions from $\mathbb{R}$ to itself. Addition and multiplication of functions in S are defined *pointwise*, so that if $f, g \in S$, then $f + g$ is the function

$$(f + g)(x) = f(x) + g(x)$$

and fg is the function

$$(fg)(x) = f(x)g(x).$$

(a) Prove that S is a ring under these operations.

(b) Does S have an identity element? Is S commutative?

(c) Let $a \in \mathbb{R}$. Define a function

$$\begin{aligned} \varphi_a\colon S &\to \mathbb{R} \\ f &\mapsto f(a). \end{aligned}$$

That is, φ_a is the **evaluation at** a function. Prove that φ_a is a ring homomorphism.

(d) Describe the kernel $\ker \varphi_a$.

2

(a) Describe all *group* homomorphisms $\varphi\colon \mathbb{Z} \to \mathbb{Z}$.

(b) Describe all *ring* homomorphisms $\varphi\colon \mathbb{Z} \to \mathbb{Z}$.

[HINT: In both cases, what can $\varphi(1)$ be?]

3 (Judson 16.28) A ring R is called a **Boolean ring** if $a^2 = a$ for every $a \in R$. Prove that every Boolean ring is commutative.

4 Let $\mathbb{Z}[x]$ be the ring of polynomials in the variable x with integer coefficients. Determine which of the following are ideals in $\mathbb{Z}[x]$. Provide justification for your answers.

(a) The set of all polynomials whose constant term is odd.

(b) The set of all polynomials whose constant term is even.

(c) The set of all polynomials whose coefficient of x^2 is even.

(d) The set of all polynomials whose constant term, coefficient of x, and coefficient of x^2 are zero.

(e) The set of all polynomials $p(x)$ such that $p'(0) = 0$, where $p'(x)$ is the usual first derivative of $p(x)$ with respect to x.

5 Let E and F be fields, and let $\varphi\colon E \to F$ be a homomorphism.

(a) Prove that if $\varphi(1) = 0$, then the image $\varphi(E)$ is equal to $\{0\}$.

(b) Prove that if $\varphi(1) \neq 0$, then $\varphi(1) = 1$.

(c) Prove that if $\varphi(1) \neq 0$, then φ is injective.