1 (Judson 3.10 – modified) Prove that the set of matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix},$$

where $x, y, z \in \mathbb{R}$, is a group under matrix multiplication. This group, known as the **Heisenberg group**, is important in quantum physics.

NOTE: The group operation is ordinary matrix multiplication, so that

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x' & y' \\ 0 & 1 & z' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+x' & y+y'+xz' \\ 0 & 1 & z+z' \\ 0 & 0 & 1 \end{pmatrix}.$$

2 (Judson 3.31) Let G be a group with the property that $g^2 = e$ for every $g \in G$. Prove that G is abelian. [Here, $e \in G$ is the identity.]

3 Let G be a group and let $x, y \in G$. Prove that the following statements are equivalent:

(a) $xy = yx$.

(b) $xyx^{-1} = y$.

(c) $xyx^{-1}y^{-1} = e$, where $e \in G$ is the identity.

NOTE: To prove this three-way equivalence, you'll need to prove several one-way implications. The most straightforward approach is to prove (a) $\Rightarrow$ (b), then (b) $\Rightarrow$ (c), and lastly (c) $\Rightarrow$ (a).