1 Let G be a group, and let $x, g \in G$.

(a) Prove by induction that $(gxg^{-1})^n = gx^n g^{-1}$ for every $n \in \mathbb{N}$.

(b) Use the result of part (a) to prove that $|gxg^{-1}| = |x|$.
[HINT: There are two cases: Either $|x| < \infty$ or $|x| = \infty$.]

(c) Deduce that $|ab| = |ba|$ for all $a, b \in G$.

2 Let

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; a, b, d \in \mathbb{R} \text{ and } ad \neq 0 \right\}.$$

Prove that H is a subgroup of $\mathrm{GL}_2(\mathbb{R})$.

3 (Judson 3.54) Let G be a group and $H \leq G$ a subgroup. For a fixed $g \in G$, define

$$gHg^{-1} = \{ghg^{-1} \mid h \in H\}.$$

This is called the **conjugate of** H **by** g**.** Prove that gHg^{-1} is also a subgroup of G.

4 Let G be a finite group with $|G| = n > 2$. Prove that G cannot have a subgroup H with $|H| = n - 1$.