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1 Let G be a group and $a, b \in G$. Show that $(ab)^{-1} = b^{-1}a^{-1}$.

Proof. We must show that

$$(ab) \cdot (b^{-1}a^{-1}) = e = (b^{-1}a^{-1}) \cdot (ab).$$

By associativity and the definition of inverses, we have

$$(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = aea^{-1} = aa^{-1} = e.$$

Similarly,

$$(b^{-1}a^{-1})(ab) = b^{-1}(a^{-1}a)b = b^{-1}eb = b^{-1}b = e,$$

as desired. □

2 Give an example of a group G for which the set

$$T = \{g \in G \mid |g| < \infty\}$$

of **torsion elements** is not a subgroup.

Solution. Let $G = \mathrm{GL}_2(\mathbb{R})$ be the group of invertible 2×2 matrices with entries in $\mathbb{R}$. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

It is easily checked that $A^2 = I_2$ and $B^2 = I_2$, and therefore $A^{-1} = A$ and $B^{-1} = B$. This means that $A, B \in \mathrm{GL}_2(\mathbb{R})$ and, moreover, $|A| = |B| = 2$, so that $A, B \in T$. However,

$$AB = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

and hence

$$(AB)^n = \begin{pmatrix} 1 & (-1)^n \\ 0 & 1 \end{pmatrix}$$

is not equal to I_2 for any integer $n > 0$. Thus $AB \notin T$, showing that T is not closed under multiplication. Therefore T is not a subgroup of $\mathrm{GL}_2(\mathbb{R})$. □