The Symmetric Group

Let A be a set the set

$$S_A = \{ f : A \to A \mid f \text{ is a bijection} \}$$

is a group under $\circ$, called the <u>symmetric group on A</u>. Elements of S_A are called <u>permutations</u> of A.

When $A = \{1, 2, \ldots, n\}$, we write S_n instead of S_A.

On HW 8, you studied S_3.

<u>Ex</u>: S_1 is the trivial group, since the only bijection

$$f : \{1\} \to \{1\}$$

is the identity.

Ex: S_2 is cyclic of order 2, since the only non-identity bijection

$$f:\{1,2\} \to \{1,2\}$$

is given by $f(1)=2$ and $f(2)=1$, so $f^2 = f \circ f = id$.

Observation: For $n \geq 3$, S_n is non-abelian.

You proved this on HW 8.

Observation: $|S_n| = n!$

n	choices	for	$f(1)$
$n-1$	"	"	$f(2)$
$\vdots$			
2	"	"	$f(n-1)$
1	choice	"	$f(n)$

Ex: Let $\sigma, \tau \in S_4$ be defined by

$\sigma: \{1,2,3,4\} \to \{1,2,3,4\}$ and $\tau: \{1,2,3,4\} \to \{1,2,3,4\}$

$$\begin{aligned} 1 &\mapsto 3 \\ 2 &\mapsto 2 \\ 3 &\mapsto 4 \\ 4 &\mapsto 1 \end{aligned} \qquad\qquad \begin{aligned} 1 &\mapsto 4 \\ 2 &\mapsto 3 \\ 3 &\mapsto 2 \\ 4 &\mapsto 1 \end{aligned}$$

Then $\sigma\tau = \sigma \circ \tau$ is

$$\begin{aligned} 1 &\mapsto 4 \mapsto 1 \\ 2 &\mapsto 3 \mapsto 4 \\ 3 &\mapsto 2 \mapsto 2 \\ 4 &\mapsto 1 \mapsto 3 \end{aligned}$$

and $\tau\sigma = \tau \circ \sigma$ is

$$\begin{aligned} 1 &\mapsto 3 \mapsto 2 \\ 2 &\mapsto 2 \mapsto 3 \\ 3 &\mapsto 4 \mapsto 1 \\ 4 &\mapsto 1 \mapsto 4 \end{aligned}$$

Two-line notation

We can condense the description of σ and τ in the example above by writing

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

Then

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

where we work right-to-left, since $\sigma\tau = \sigma \circ \tau$ is a function composition.

Compute $\tau\sigma$ similarly.