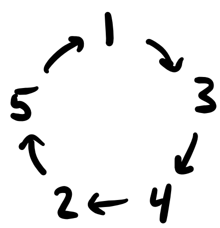
Warm-Up: Let $\sigma, \tau \in S_5$ be

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 2 & 1 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix}.$$

Compute $\sigma\tau$ and $\tau\sigma$.

Cycle notation

Notice that σ describes a "cycle":



We write this as $\sigma = (1\ 3\ 4\ 2\ 5)$.

(Also, $\sigma = (3\ 4\ 2\ 5\ 1) = (4\ 2\ 5\ 1\ 3) = \cdots$)

Similarly, τ consists of two cycles

$$\begin{tikzcd} 1 \arrow[d, bend left] \\ 4 \arrow[u, bend left] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & 2 \arrow[dr, bend left] & \\ 5 \arrow[ur, bend left] & & 3 \arrow[ll, bend left] \end{tikzcd}$$

Write $\tau = (1\ 4)(2\ 3\ 5)$.

Notice that $\tau = (2\ 3\ 5)(1\ 4)$ also, because the cycles are <u>disjoint</u>.

Now, working right-to-left, we compute

$$\sigma\tau = (1\ 3\ 4\ 2\ 5)(1\ 4)(2\ 3\ 5)$$

$$= (1\ 2\ 4\ 3)(5)$$

$1 \to 4 \to 2$
$2 \to 3 \to 4$
etc.

and

$$\tau\sigma = (1\ 4)(2\ 3\ 5)(1\ 3\ 4\ 2\ 5)$$

$$= (1\ 5\ 4\ 3)(2).$$

Note: We normally omit the 1-cycles, and understand them to be implicit.

Ex: The elements of S_3 are

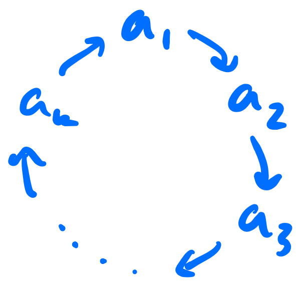
With 1-cycles	Without
$\mathrm{id} = (1)(2)(3)$	e
$(1\ 2)(3)$	$(1\ 2)$
$(1\ 3)(2)$	$(1\ 3)$
$(1)(2\ 3)$	$(2\ 3)$
$(1\ 2\ 3)$	$(1\ 2\ 3)$
$(1\ 3\ 2)$	$(1\ 3\ 2)$

Observe that $(1\ 2\ 3)^{-1} = (3\ 2\ 1)$
$= (1\ 3\ 2)$

Prop: A k-cycle in S_n has order k.

Proof: Let $\sigma = (a_1 \ a_2 \ \cdots \ a_k)$ be a k-cycle.

Visualize:



Then $\sigma^i \neq id$ for $1 \leq i \leq k-1$, since

$$\sigma^i(a_1) = a_{i+1} \neq a_1$$

for these values of i.

But $\sigma^k(a_1) = a_1$, $\sigma^k(a_2) = a_2, \ldots,$ and $\sigma^k(a_k) = a_k$, so

$$\sigma^k = id.$$

Thus, $|\sigma| = k$. $\blacksquare$

Observations:

- Every $\sigma \in S_n$ is the product of disjoint cycles (Book Thm 5.9)
- Disjoint cycles commute (Book Prop 5.8)

Prop: If $\sigma \in S_n$ is expressed as a product of disjoint cycles, then $|\sigma|$ is the least common multiple of the cycle lengths.

Proof: Let $\sigma = \tau_1 \tau_2 \cdots \tau_m$, where the τ_i are disjoint cycles. Then

$$\sigma^k = \tau_1^k \tau_2^k \cdots \tau_m^k,$$

so $|\sigma|$ is the smallest positive integer which is a multiple of every cycle length. ∎