Important facts about cycle notation:

- Disjoint cycles commute (Book Prop 5.8)

- Every permutation is a product of disjoint cycles (Book Thm 5.9), unique up to reordering.

Ex: The elements of S_4, by cycle type:

Cycle type	Permutations	#
$(1,1,1,1)$	$(1)(2)(3)(4) = e$	1
$(2,1,1)$ "2-cycles" or "transpositions"	$(1\ 2), (1\ 3), \ldots, (3\ 4)$	6
$(3,1)$ "3-cycles"	$(1\ 2\ 3), \ldots, (2\ 4\ 3)$	8
(4) "4-cycles"	$(1\ 2\ 3\ 4), \ldots, (1\ 4\ 3\ 2)$	6
$(2,2)$ "(2,2)-cycles"	$(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)$	3

total = 24 ✓

Transpositions

A 2-cycle is also called a <u>transposition</u>.

<u>HW 10</u>: Every cycle is a product of (non-disjoint!) transpositions.

<u>Ex</u>: $(1\ 2\ 3) = (1\ 2)(2\ 3)$

$= (1\ 3)(1\ 2)$

$= (1\ 2)(1\ 3)(2\ 3)(1\ 2)$

Since every permutation is a product of cycles, this means <u>every</u> permutation can be written as a product of transpositions.

We say "S_n is generated by transpositions."

As the example shows, writing $\sigma \in S_n$ as a product of transpositions is <u>not</u> unique.

However...

<u>Thm</u>: Let $\sigma \in S_n$. Then either

- Any expression of σ as a product of transpositions contains an <u>even</u> number of transpositions,

OR

- Any expression of σ as a product of transpositions contains an <u>odd</u> number of transpositions.

In the first case, we say σ is an <u>even permutation</u> and write $\text{sgn}(\sigma) = 1$.

In the second case, we say σ is an <u>odd permutation</u> and write $\text{sgn}(\sigma) = -1$.

The number $\mathrm{sgn}(\sigma) \in \{1, -1\}$ is called the <u>sign</u> or <u>signature</u> of σ.

<u>Ex</u>: We saw above that $\mathrm{sgn}((1\ 2\ 3)) = 1$, i.e., $(1\ 2\ 3)$ is an even permutation.

The sign depends only on the cycle type, so <u>every</u> 3-cycle is even.

<u>Ex</u>: S_4 again

	Cycle type	Permutations	#
Even	$(1,1,1,1)$	$(1)(2)(3)(4) = e$	1
Odd	$(2,1,1)$	$(1\ 2), (1\ 3), \ldots, (3\ 4)$	6
Even	$(3,1)$	$(1\ 2\ 3), \ldots, (2\ 4\ 3)$	8
Odd	(4)	$(1\ 2\ 3\ 4), \ldots, (1\ 4\ 3\ 2)$	6
Even	$(2,2)$	$(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)$	3

We will prove the theorem on Wednesday.

Dihedral groups

Def: Let $n \geq 3$. The dihedral group is the group of symmetries of a regular n-gon.

Notes:

- Regular = all angles equal / all sides same length
  $\triangle$, $\square$, ⬠, etc.
- Symmetry = rigid motion preserving the shape
- Group operation = composition

What are the elements of D_n?

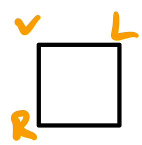
Pick a vertex v.

A symmetry must map v to one of the n vertices.

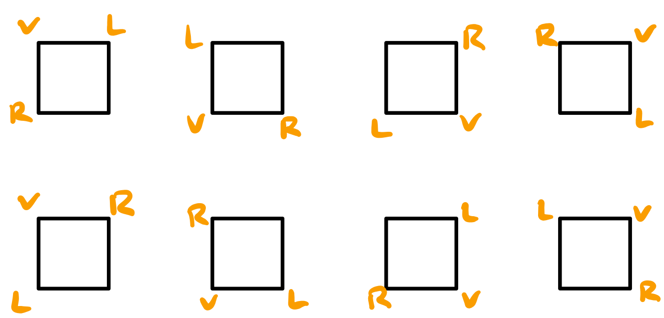
Its neighbors must come along, and can be in one of two orientations.

Once we know where v and its neighbors are, we know where every vertex is.

So $\boxed{|D_n| = 2n}$.

Ex: D_4 The 8 symmetries map 

to one of



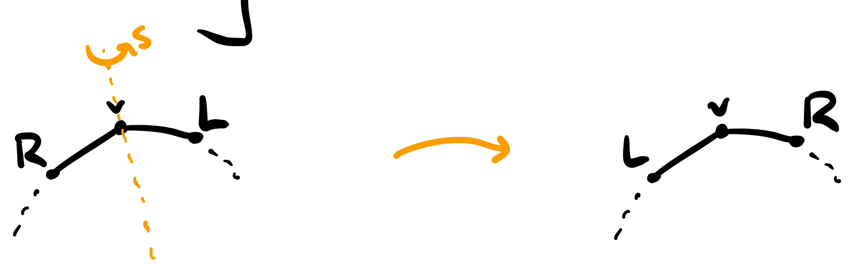
Let r be the counterclockwise rotation by $\frac{2\pi}{n}$ radians.

Then $|r| = n$, so $\langle r \rangle = \{e, r, r^2, \dots, r^{n-1}\}$ is a cyclic subgroup containing half the elements in D_n.

To find the other n, let s be the reflection fixing vertex v.

<u>Claim</u>: The other n elements of D_n are $s, sr, sr^2, \dots, sr^{n-1}$.

<u>Proof</u>: First, note that $s \neq r^i$ for any i, since r^i preserves the ordering of v's neighbors and s switches this ordering.



Now, suppose $sr^i = sr^j$. By cancellation, $r^i = r^j$, so $i \equiv j \pmod{n}$.

In particular, $s, sr, \ldots, sr^{n-1}$ are pairwise distinct, and none is in $\langle r \rangle$. $\blacksquare$

So we now know

$$D_n = \{e, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}\}.$$