We have the presentation

$$D_n = \langle \underbrace{r, s}_{\text{Generators}} \mid \underbrace{r^n = s^2 = e, \quad rs = sr^{-1}}_{\text{Relations}} \rangle.$$

This means:

① Every element in D_n is obtained as a product involving r's and s's

We proved $D_n = \{e, r, r^2, \dots, r^{n-1}, s, sr, \dots, sr^{n-1}\}$

② Every equality of elements in D_n can be deduced from the given relations.

They allow us to write any product in the standard form, $s^i r^j$ for $0 \le i \le 1$, $0 \le j \le n-1$.

Warm-Up: In D_5, write the following elements in the form $s^i r^j$.

i) $r^4 srs$ ii) $(r^3 s^3)^{-1}$ iii) $s^3 rsr^3 s^2 r^4 s$

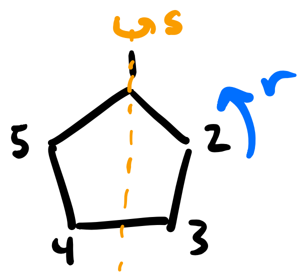
Permutation Representation of D_n

Here is how the book introduces D_n.

Label the vertices of a regular n-gon as $1, 2, \dots, n$.

Then r and s each induce permutations of $\{1, 2, \dots, n\}$, so they correspond to elements in S_n. They generate a subgroup of S_n isomorphic to D_n.

Ex: $n = 5$



Then

$$r = (1\ 5\ 4\ 3\ 2)$$
$$s = (2\ 5)(3\ 4),$$

and it's easy to check that $r^5 = s^2 = e$ and $rs = (1\,5)(2\,4) = sr^{-1}$.

Generators and relations

In general, a set $S \subseteq G$ <u>generates</u> G if every element in G can be written as a product of elements in S (and their inverses).

A <u>relation</u> in G is an equation involving elements of G.

A <u>presentation</u> of G consists of a generating set S and a set of relations R such that every relation in G is implied by R.

Write $G = \langle S \mid R \rangle$.

Ex: The *quaternion group* is

$$Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$$

where

- 1 is the identity
- -1 multiplies the way you think it should (e.g., $(-1)i = -i$, $(-1)j = -j$, etc.)
- $i^2 = j^2 = k^2 = -1$
- $ij = k$, $jk = i$, $ki = j$
- $ji = -k$, $kj = -i$, $ik = -j$

Thm: Q_8 is a group.

Proof: Check associativity (easy but tedious).

A presentation allows us to define Q_8 more compactly. Here is one:

$$Q_8 = \langle -1, i, j, k \mid (-1)^2 = 1,\ i^2 = j^2 = k^2 = ijk = -1 \rangle.$$

It takes work to show this is correct!

For example, we can deduce $ij = k$ from

$$ijk = -1 = k^2$$

so by cancellation $ij = k$.

Here is a more compact presentation:

$$Q_8 = \langle i, j \mid i^4 = 1,\ i^2 = j^2,\ ji = i^{-1}j \rangle.$$

Again, it takes work to show this is the same group!

Ex: A presentation for S_n.

Let $\tau_i = (i \;\; i+1)$. Then

$$S_n = \left\langle \tau_1, \tau_2, \ldots, \tau_{n-1} \;\middle|\; \begin{array}{l} \cdot\ \tau_i^2 = e \\ \cdot\ \tau_i \tau_j = \tau_j \tau_i \text{ if } j \notin \{i-1, i, i\} \\ \cdot\ \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \end{array} \right\rangle$$

This is not obvious, and takes work to prove!

Presentations are not always useful, and can obscure the nature of the group.

Ex: Let $G = \langle a, b \mid aba = bab,\ (aba)^3 = e \rangle$.

It can be proved that $a = b$ and $G = \langle a \rangle$ is cyclic of order 9.

Ex: Let $H = \langle a, b \mid aba = bab,\ (aba)^4 = e \rangle$.

It can be proved that H is an infinite non-abelian group!

In fact, $H \cong \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \middle| x, y, z, w \in \mathbb{Z} \text{ and } xw - yz = 1 \right\}$

via

$$a \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$b \mapsto \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

so then

$$aba = bab \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$