Warm-Up: Write the following as a product of "adjacent transpositions" $(i\ i{+}1)$

- $(1\ 3)$
- $(2\ 5)$
- $(3\ 8)$

I owe you a proof of:

Thm: The sign $\mathrm{sgn}(\sigma)$ of a permutation $\sigma \in S_n$ is well-defined.

Recall:
- $\mathrm{sgn}(\sigma) = 1$ if σ is the product of an <u>even</u> number of transpositions.
- $\mathrm{sgn}(\sigma) = -1$ if σ is the product of an <u>odd</u> number of transpositions

The key step is to prove:

Lemma: The identity permutation is even.
That is, if $\tau_1, \dots, \tau_r$ are transpositions such that

$$\tau_1 \tau_2 \cdots \tau_r = \mathrm{id},$$ (Note: $r=0$ is OK)

then r must be even.

Proof of Theorem (assuming the Lemma):

Suppose
$$\sigma = \tau_1 \cdots \tau_k$$
$$= \mu_1 \cdots \mu_\ell,$$

where the τ_i and μ_j are transpositions.
Then
$$\sigma^{-1} = \tau_k \cdots \tau_1,$$
so
$$\mathrm{id} = \sigma^{-1}\sigma = \tau_k \cdots \tau_1 \mu_1 \cdots \mu_\ell.$$

By the Lemma, $k+l$ must be even, so k and l are either both even or both odd.

$\boxtimes$

Proof #1 of Lemma (Book's method):

Consider
$$\tau_1 \cdots \tau_{r-1} \tau_r = id.$$

If $r=0$, there is nothing to prove. Since the identity is not a transposition, $r \neq 1$.

So assume $r \geq 2$. Consider $\tau_r = (a\ b)$. We consider cases for what τ_{r-1} could be:

Case 1: $\tau_{r-1} = \tau_r = (a\ b)$.

Then $\tau_{r-1}\tau_r = (a\ b)^2 = id$, so

$$id = \tau_1 \cdots \tau_{r-2}$$

$\underline{\text{Case 2}}$: $T_{r-1} = (a \;\; c)$ for some $c \neq b$.

Then $T_{r-1} T_r = (a \;\; c)(a \;\; b)$

$= (\underline{a} \;\; b)(b \;\; c)$.

$\underline{\text{Case 3}}$: $T_{r-1} = (b \;\; c)$ for some $c \neq a$

Then $T_{r-1} T_r = (b \;\; c)(a \;\; b)$

$= (\underline{a} \;\; c)(b \;\; c)$

$\underline{\text{Case 4}}$: $T_{r-1} = (c \;\; d)$ where $c \neq a, b$
$d \neq a, b$

Then $T_{r-1} T_r = (c \;\; d)(a \;\; b)$

$= (\underline{a} \;\; b)(c \;\; d)$

So either we

- Decrease the number of transpositions by 2 (Case 1)

OR

- Move the first instance of a one spot to the left (Cases 2-4)

Continue in this fashion, moving the first instance of a one spot to the left until we get a cancellation.

We must eventually get a cancellation.

If not, then id is a product of transpositions where a only occurs in the first transposition.

But such a product doesn't fix a, so is not equal to id.

Thus, if id is equal to a product of $r>1$ transpositions, it is also equal to a product of $r-2$ transpositions.

By induction, r is even. ∎

Here's another approach:

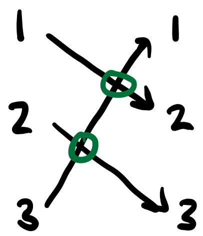
Def: Let $\sigma \in S_n$. An <u>inversion</u> of σ is a pair (i,j) such that $i<j$ but $\sigma(i) > \sigma(j)$.

Ex: Consider $\sigma = (1\ 2\ 3) \in S_3$.

i	j	$\sigma(i)$	$\sigma(j)$	inversion?
1	2	2	3	N
1	3	2	1	Y
2	3	3	1	Y

So σ has <u>two</u> inversions.

We can visualize this as



Crossings in this diagram correspond to inversions (Prove it!)

Claim: Suppose $\tau_1, \ldots, \tau_k \in S_n$ are transpositions, and let $\sigma = \tau_1 \cdots \tau_k$. Then

- If k is odd, then σ has an odd number of inversions.
- If k is even, then σ has an even number of inversions.

Proof #2 of Lemma (assuming the Claim):

We have that id has 0 inversions, and 0 is even.

$\blacksquare$

So we just need to prove the claim.

Proof of Claim (sketch):

Step 1: Show that any transposition is the product of an odd number of "adjacent" transpositions $(i \;\; i+1)$.

Basically done in Warm-Up.

This means that any product $\tau_1 \cdots \tau_k$ of transpositions can be replaced by a longer product of adjacent transpositions, where the number of adjacent transpositions is odd if and only if k is odd (and so even if and only if k is even).

<u>Step 2</u>: Argue that multiplication by an adjacent transposition changes the number of inversions by ± 1.

<u>Idea</u>: σ and $\sigma \cdot (i\ i{+}1)$ have exactly the same inversions, except $(i, i{+}1)$ is an inversion of one but not the other.

Together, these 2 steps prove the Claim. $\blacksquare$