More set theory review

Let A and B be sets.

- The <u>intersection</u> of A and B is

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

- The <u>union</u> of A and B is

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

- The <u>set difference</u> or <u>relative complement</u> is

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$$

- If all the objects we are interested in come from some set U ("universe"), then the <u>complement</u> of $A \subseteq U$ is

$$A^c = U \setminus A.$$

<u>Caution</u>: The textbook uses

- $A \subset B$ instead of $A \subseteq B$
- A' instead of A^c

<u>Thm</u> (De Morgan's laws for sets).

Let A and B be sets, each a subset of a universe $\mathcal{U}$. Then

① $(A \cup B)^c = A^c \cap B^c$

② $(A \cap B)^c = A^c \cup B^c$

<u>Proof</u>: Math 3345 / text.

The <u>Cartesian product</u> of two sets A and B is

$$A \times B = \{ (x,y) \mid x \in A \text{ and } y \in B \}.$$

A <u>function</u> (or <u>mapping</u> or <u>map</u>) from a set A to a set B is a subset

$$f \subseteq A \times B$$

such that

(*) for every $x \in A$, there is a unique $y \in B$ such that $(x,y) \in f$.

We never use this notation! Write

- $f : A \to B$ instead of $f \subseteq A \times B$
- $f(x) = y$ instead of $(x,y) \in f$.

For a function $f : A \to B$,

- the <u>domain</u> is A,
- the <u>codomain</u> is B,
- the <u>range</u> (or <u>image</u>) is

$$f(A) = \{ f(x) \mid x \in A \}.$$

The function $f : A \to B$ is <u>injective</u> (or <u>one-to-one</u>) if for all $x_1, x_2 \in A$,

$$f(x_1) = f(x_2) \implies x_1 = x_2.$$

[Equivalently, $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$.]

The function $f : A \to B$ is <u>surjective</u> (or <u>onto</u>) if $f(A) = B$.

<u>Note</u>: $f(A) \subseteq B$ by definition.

A function is <u>bijective</u> if it is both injective and surjective.

Nouns: Injection, surjection, bijection.

On any set S there is the <u>identity function</u>

$$\begin{aligned} id_S : S &\to S \\ x &\mapsto x. \end{aligned}$$

If $f: A \to B$ is a bijection, then for each $y \in B$ <u>there exists</u> a <u>unique</u> $x \in A$ such that $f(x) = y$.

surjective injective

This property defines the <u>inverse function</u> $f^{-1}: B \to A$.

Properties (from Math 3345)

Suppose $f: A \to B$ is a bijection.
Then

- $f^{-1} \circ f = id_A : A \to A$.
- $f \circ f^{-1} = id_B : B \to B$.
- f^{-1} is also a bijection, with $(f^{-1})^{-1} = f$.

Thm: Let $f: A \to B$ be a function.
If there exists $g: B \to A$ such that

$$g \circ f = id_A \quad \text{and} \quad f \circ g = id_B,$$

then f is a bijection and $g = f^{-1}$.

Equivalence relations

Def: A relation R on a set S is a subset of $S \times S$

Notation: Instead of writing $(x,y) \in R$, we usually write $x R y$.

Ex:
- $\leq$ is a relation on $\mathbb{R}$ (or $\mathbb{Z}$, or $\mathbb{Q}$)
- So is $<$.
- $=$ is a relation on any set
- $\neq$ is a relation on any set

Def: A relation $\sim$ on a set S is an equivalence relation if for all $x, y, z \in S$,

1. $x \sim x$, [reflexive]
2. if $x \sim y$, then $y \sim x$, [symmetric]
3. if $x \sim y$ and $y \sim z$, then $x \sim z$. [transitive]

<u>Ex</u>:
- $=$ is an equivalence relation
- $\leq$ fails ②
- $<$ fails ① and ②
- $\neq$ fails ① and ③