Warm-Up: Let G be a group, and let $H \leq G$ be a subgroup.

Define a relation $\sim_L$ on G by "Left"

$x \sim_L y$ if and only if $x^{-1}y \in H$.

Two extreme examples:

- If $H = \{e\}$, then $x \sim_L y \iff x^{-1}y = e$
  $\iff x = y$.

- If $H = G$, then $x \sim_L y$ for all $x, y \in G$.

Prove that $\sim_L$ is an equivalence relation.

Cosets

Since $\sim_L$ is an equivalence relation, it partitions G into equivalence classes. What are they?

For $g \in G$ and $x \in G$, we have

$$\begin{aligned} x \in [g] &\iff g \sim_L x \\ &\iff g^{-1}x \in H \\ &\iff g^{-1}x = h \text{ for some } h \in H \\ &\iff x = gh \text{ for some } h \in H. \end{aligned}$$

Def: Let G be a group, $H \leq G$ a subgroup, and $g \in G$.
The left coset of H in G containing g is

$$gH := \{gh \mid h \in H\}$$

That is, $gH = [g]$ is the equivalence class containing g for $\sim_L$.

Thus, the left cosets of H in G partition G

Each $g \in G$ is in exactly one left coset, namely gH.

Note: $eH = H$ is one of the cosets, and the only one which is a group.

Ex: $G = S_3$, $H = \langle (1\,2) \rangle = \{e, (12)\}$.

Left cosets of H in G

- $H = \{e, (1\,2)\} = eH = (1\,2)H$
- $(1\,3)H = \{(1\,3), (1\,2\,3)\} = (123)H$
- $(2\,3)H = \{(2\,3), (1\,3\,2)\} = (1\,3\,2)H$

Ex: $G = S_3$, $K = \langle (1\,2\,3) \rangle = A_3$

Left cosets of K in G

- $K = \{e, (1\,2\,3), (1\,3\,2)\} = (1\,2\,3)K = (1\,3\,2)K$
- $(1\,2)K = \{(1\,2), (2\,3), (1\,3)\} = (2\,3)K = (1\,3)K$

Remark: We can repeat this entire process starting with the relation

$$x \sim_R y \quad \text{if and only if} \quad xy^{-1} \in H.$$

"Right"

Then

- $\sim_R$ is an equivalence relation.
- The equivalence classes are <u>right cosets</u>
  $$Hg := \{hg \mid h \in H\}.$$
- G is therefore also partitioned into right cosets.

Ex: $G = S_3$, $H = \langle (1\,2) \rangle$, $K = \langle (1\,2\,3) \rangle$

<u>Right cosets of H in G</u>

- $H = \{e, (1\,2)\} = H(1\,2)$

- $H(1\,3) = \{(1\,3), (1\,3\,2)\} = H(1\,3\,2)$

- $H(2\,3) = \{(2\,3), (1\,2\,3)\} = H(1\,2\,3)$

<u>Right cosets of K in G</u>

- $K = \{e, (1\,2\,3), (1\,3\,2)\} = K(1\,2\,3) = K(1\,3\,2)$

- $K(1\,2) = \{(1\,2), (1\,3), (2\,3)\} = K(1\,3) = K(2\,3)$

So the partition into right cosets <u>may or may not</u> be the same as the partition into left cosets

Lemma: Let G be a group and $H \leq G$ a subgroup. Then for any $g \in G$,

$$|gH| = |Hg| = |H|.$$

Proof: Define a function

$$\varphi : H \to gH$$
$$h \mapsto gh.$$

Then φ is surjective, since by definition each element of gH is

$$gh = \varphi(h)$$

for some $h \in H$.

To see φ is also injective, suppose $\varphi(h_1) = \varphi(h_2)$ for some $h_1, h_2 \in H$.

Then $gh_1 = gh_2$, so $h_1 = h_2$ by cancellation.

Thus, φ is a bijection and so $|H| = |gH|$.

The proof of $|H| = |Hg|$ is similar.

$\blacksquare$

Def: Let G be a group and $H \leq G$ a subgroup.

The <u>index of H in G</u> is the number of distinct left cosets of H in G, and is denoted by $[G : H]$.

We will soon see that $[G:H]$ is <u>also</u> the number of distinct right cosets.