Last time, we proved

Lemma: Let $\varphi : G \to H$ be a group homomorphism. Then

① $\varphi(e_G) = e_H$, where e_G is the identity of G and e_H is the identity of H.

② For all $g \in G$, $\varphi(g^{-1}) = (\varphi(g))^{-1}$.

③ For all $g \in G$ and $n \in \mathbb{Z}$,

$$\varphi(g^n) = (\varphi(g))^n.$$

④ If $K \leq G$ is a subgroup of G, then $\varphi(K) = \{\varphi(k) \mid k \in K\}$ is a subgroup of H.

In particular, the range $\varphi(G) \leq H$.

I want to examine some consequences of this Lemma.

Thm: Let $\varphi : G \to H$ be an isomorphism.
Then

1. For all $g \in G$, $|g| = |\varphi(g)|$.

2. The subgroup lattice of H is "the same" as the subgroup lattice of G – just apply φ to every subgroup of G.

Technically, φ induces a lattice isomorphism.

A structural property of a group is a property preserved by isomorphism.

The theorem says that

- the orders of elements

and

- the subgroup lattice

are structural properties.

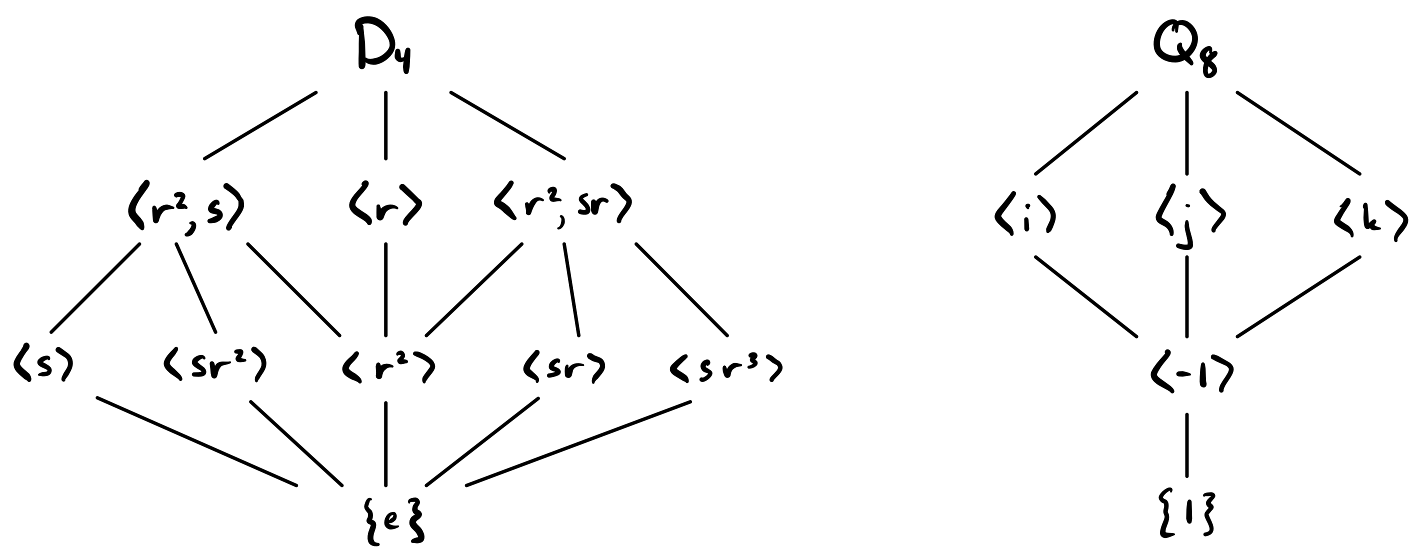
Useful to show groups aren't isomorphic.

Ex: $D_4 \not\cong Q_8$.

Proof 1: D_4 has 5 elements of order 2 (s, sr, sr^2, sr^3, r^2) and 2 elements of order 4 (r, r^3).

Q_8 has 1 element of order 2 (-1) and 6 elements of order 4 $(\pm i, \pm j, \pm k)$. $\blacksquare$

Proof 2: Look at the subgroup lattices.



$\blacksquare$

Thm: If $\varphi : G \to H$ is an injective group homomorphism, then $G \cong \varphi(G)$.

(So G is isomorphic to a <u>subgroup</u>, $\varphi(G)$, of H.)

Proof: This is essentially automatic. The only thing missing for φ to be an isomorphism is surjectivity, and φ is surjective onto $\varphi(G)$. $\blacksquare$