Last time: The *direct product* of groups G and H is $G \times H$, where the operation is

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2).$$

Observe: $G \times H \cong H \times G$ (why?)

Warm-Up: Describe the groups $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_3$.

Takeaway:
- a product of cyclic groups can be cyclic ($\mathbb{Z}_2 \times \mathbb{Z}_3$)
- but it doesn't have to be ($\mathbb{Z}_2 \times \mathbb{Z}_2$)

Lemma: Let G and H be groups, with $g \in G$ and $h \in H$.

If $|g| = \infty$ or $|h| = \infty$, then $|(g,h)| = \infty$.

Otherwise, if $|g|$ and $|h|$ are finite, then
$$|(g,h)| = \operatorname{lcm}(|g|, |h|).$$

Proof: Let $k \in \mathbb{N}$. Then
$$(g,h)^k = (g^k, h^k).$$

So $|(g,h)|$, if it exists, is the smallest $k \in \mathbb{N}$ such that both

- $g^k = e_G$ $\Leftrightarrow$ k is a multiple of $|g|$
- $h^k = e_H$ $\Leftrightarrow$ k is a multiple of $|h|$

$\blacksquare$

Thm: Let $n, m \in \mathbb{N}$. Then

$$\mathbb{Z}_n \times \mathbb{Z}_m \text{ is cyclic } \Leftrightarrow \gcd(n,m) = 1,$$

in which case $\mathbb{Z}_n \times \mathbb{Z}_m \cong \mathbb{Z}_{nm}$.

Proof: Since $|\mathbb{Z}_n \times \mathbb{Z}_m| = nm$, the only cyclic group $\mathbb{Z}_n \times \mathbb{Z}_m$ could be isomorphic to is $\mathbb{Z}_{nm}$.

Recall: $\operatorname{lcm}(n,m) = \dfrac{nm}{\gcd(n,m)}$

($\Leftarrow$) Suppose $\gcd(n,m) = 1$. Then, since $1 \in \mathbb{Z}_n$ has order n and $1 \in \mathbb{Z}_m$ has order m, the element $(1,1) \in \mathbb{Z}_n \times \mathbb{Z}_m$ has order

$$\operatorname{lcm}(n,m) = \frac{nm}{1} = nm$$

by the lemma. Thus, $\mathbb{Z}_n \times \mathbb{Z}_m = \langle (1,1) \rangle$.

($\Longrightarrow$) By contrapositive.

Suppose $\gcd(n,m) > 1$.
Then

$$\text{lcm}(n,m) = \frac{nm}{\gcd(n,m)} < nm.$$

Now, for any $(a,b) \in \mathbb{Z}_n \times \mathbb{Z}_m$,

- $|a|$ divides n
- $|b|$ divides m
- both n and m divide $\text{lcm}(n,m)$.

Hence,

$$\underbrace{(a,b) + \cdots + (a,b)}_{\text{lcm}(n,m) \text{ times}} = (0,0),$$

so $|(a,b)| \leq \text{lcm}(n,m) < nm.$

That is, $\mathbb{Z}_n \times \mathbb{Z}_m$ has no elements of order nm, so it is not cyclic. $\blacksquare$

More generally, we can make the product of many groups:

$$\prod_{i=1}^{k} G_i = G_1 \times G_2 \times \cdots \times G_k .$$

<u>Cor</u>: Let $n_1, \ldots, n_k \in \mathbb{N}$. Then

$$\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k} \cong \mathbb{Z}_{n_1 n_2 \cdots n_k}$$

if and only if the numbers $n_1, \ldots, n_k$ are <u>pairwise relatively prime</u>, meaning $\gcd(n_i, n_j) = 1$ when $i \neq j$.

<u>Proof</u>: Induct on k and use the theorem. $\blacksquare$

Ex: $\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_5 \cong \mathbb{Z}_{12} \times \mathbb{Z}_5 \cong \mathbb{Z}_{60}$

Ex: $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \cong \mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_5$

$$\cong \mathbb{Z}_2 \times \mathbb{Z}_{30}.$$

So we now have a "recipe" for finding abelian groups of a given order.

Ex: $\mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

are pairwise non-isomorphic abelian groups of order 8.

It turns out that, up to isomorphism, these are <u>all</u> abelian groups of order 8.

Thm (Fundamental Theorem of Finite Abelian Groups):

If G is an abelian group, then there exist

- primes $p_1, \dots, p_k$ (not necessarily distinct)
- integers $a_1, \dots, a_k \in \mathbb{N}$

such that

$$G \cong \mathbb{Z}_{p_1^{a_1}} \times \cdots \times \mathbb{Z}_{p_k^{a_k}}.$$

Moreover, this expression is unique up to reordering the factors.

Proof: In Judson Ch. 13.

Uses some ideas we haven't seen yet. ∎

$\underline{\text{Ex}}$: Let's find all abelian groups of order $180 = 2^2 \cdot 3^2 \cdot 5$.

- $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
- $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
- $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
- $\mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5 \cong \mathbb{Z}_{180}$