External vs. Internal Direct Products

The direct products we've seen so far are "external," in that we start with 2 groups, and produce a new group "outside" of either.

The notion of an "internal" direct product asks the inverse problem:

Given a group G, when can we decompose it as $G \cong H \times K$?

Ex: $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$.

$\mathbb{Z}_6$ also has subgroups

$$\langle 3 \rangle = \{0, 3\} \cong \mathbb{Z}_2$$
$$\langle 2 \rangle = \{0, 2, 4\} \cong \mathbb{Z}_3.$$

And we get an isomorphism

$$\mathbb{Z}_6 \longrightarrow \langle 3 \rangle \times \langle 2 \rangle$$

$0 \longmapsto (0, 0)$ $\quad 0 + 0 = 0$

$1 \longmapsto (3, 4)$ $\quad 3 + 4 = 1$

$2 \longmapsto (0, 2)$ $\quad 0 + 2 = 2$

$3 \longmapsto (3, 0)$ $\quad 3 + 0 = 3$

$4 \longmapsto (0, 4)$ $\quad 0 + 4 = 4$

$5 \longmapsto (3, 2)$ $\quad 3 + 2 = 5$

Def: Let G be a group, with subgroups H and K.

We say G is the **internal direct product** of H and K if

- $G = HK = \{hk \mid h \in H, k \in K\}$,
- $H \cap K = \{e\}$,
- $hk = kh$ for all $k \in K$ and $h \in H$.

Thm: Let G be the internal direct product of subgroups H and K. Then $G \cong H \times K$.

Idea: Each $g \in G$ can be expressed uniquely as $g = hk$ for some $h \in H$ and $k \in K$, just like we found for $\mathbb{Z}_6$.

Proof: Let $g \in G$. Since $G = HK$, we can write

$$g = hk \qquad (\star)$$

for some $h \in H$, $k \in K$.

Suppose $g = h_1 k_1 = h_2 k_2$ for some $h_1, h_2 \in H$ and $k_1, k_2 \in K$. Then

$$h_2^{-1} h_1 = k_2 k_1^{-1} \in H \cap K.$$

But $H \cap K = \{e\}$, so $h_2^{-1} h_1 = e = k_2 k_1^{-1}$, whence $h_1 = h_2$ and $k_1 = k_2$.

So the expression in $(\star)$ is unique. Thus, the map

$$\varphi : H \times K \to G$$
$$(h, k) \mapsto hk$$

is a bijection.

It is a homomorphism, since

$$
\begin{aligned}
\varphi\big((h_1, k_1)\cdot(h_2, k_2)\big) &= \varphi(h_1h_2, k_1k_2) \\
&= h_1 \underline{h_2 k_1} k_2 \qquad\qquad h_2k_1 = k_1h_2 \\
&= h_1k_1\, h_2k_2 \\
&= \varphi(h_1, k_1)\cdot\varphi(h_2, k_2).
\end{aligned}
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Normal Subgroups

Def: Let G be a group and $H \leq G$ a subgroup. We say H is a normal subgroup if

$$gH = Hg$$

"the left and right cosets are the same"

for all $g \in G$.

Write $H \trianglelefteq G$ to indicate H is normal.