Recall: A **normal subgroup** of G is a subgroup $H \leq G$ such that

$$gH = Hg$$

for every $g \in G$.

Notation: $H \trianglelefteq G$

Warm-Up: Show that in D_4

- $\langle r \rangle$ is normal,
- $\langle s \rangle$ is not.

Ex: If G is abelian, then any subgroup $H \leq G$ is normal, since

$$gH = \{\underbrace{gh}_{=hg} \mid h \in H\} = \{hg \mid h \in H\} = Hg.$$

HW14: If $H \leq G$ with $[G:H] = 2$, then $H \trianglelefteq G$.

So

- $\langle r \rangle$ is normal in D_n
- A_n is normal in S_n
- $\langle i \rangle$, $\langle j \rangle$, and $\langle k \rangle$ are normal in Q_8.

Thm: Let G be a group and $N \leq G$ a subgroup.

TFAE: (The following are equivalent)

① $N \trianglelefteq G$ (i.e. $gN = Ng$ for all $g \in G$).

② $gNg^{-1} \subseteq N$ for all $g \in G$.

③ $gNg^{-1} = N$ for all $g \in G$.

Proof: ① $\Rightarrow$ ②: Let $g \in G$ and $n \in N$.

Since

$$gn \in gN \underset{N \trianglelefteq G}{=} Ng,$$

there exists some $n' \in N$ such that $gn = n'g$. Hence

$$gng^{-1} = n' \in N.$$

That is,

$$gNg^{-1} = \{gng^{-1} \mid n \in N\} \subseteq N.$$

②⇒③: Let $g \in G$. We must prove the reverse containment,

$$N \subseteq gNg^{-1}.$$

So let $n \in N$. Then

$$g^{-1}ng = g^{-1}n(g^{-1})^{-1} \in g^{-1}N(g^{-1})^{-1}$$

But $g^{-1}N(g^{-1})^{-1} = N$, so

$$g^{-1}ng = n' \in N.$$

Thus,

$$n = gn'g^{-1} \in gNg^{-1},$$

so $N \subseteq gNg^{-1}$.

③ ⇒ ①: Let $g \in G$. Since $gNg^{-1} = N$, for any $n \in N$ there exists $n' \in N$ such that

$$gng^{-1} = n'.$$

So

$$gn = n'g \in Ng,$$

proving

$$gN \subseteq Ng.$$

To get $Ng \subseteq gN$, start with $g^{-1}Ng = N$ and repeat the same argument.

$\blacksquare$

Why normal subgroups?

Let G be a group and $H \leq G$ a subgroup. Write

$$G/H = \{gH \mid g \in G\}$$

for the set of all left cosets of H in G.

Bold Claim: G/H "should" be a group, under the operation

$$(g_1H)(g_2H) = (g_1g_2)H.$$

Problem: This is not always well-defined

Ex: Let $G = D_4$, $H = \langle s \rangle$.

Then we want

$$(rH)(r^3H) = r^4H = H \qquad (r^4 = e \in H)$$

But $rH = sr^3H$ $\left(\text{since } (sr^3)^{-1}r = (sr^3)r = s \in H\right)$, so we also have

$$(rH)(r^3H) = (sr^3H)(r^3H)$$
$$= sr^2H.$$

However, $sr^2H \neq H$!

So the "product" $(g_1H)(g_2H)$ seems to depend on the choice of coset representatives (g_1 and g_2).

It turns out that normality fixes this.

Thm: Let G be a group and $H \leq G$.

Coset multiplication in G/H is well-defined as

$$(aH)(bH) = abH$$

if and only if $H \trianglelefteq G$.

Proof: ($\Rightarrow$) Assume coset multiplication is well-defined.

Let $g \in G$. We must show $gH = Hg$.

We have

$$(gH)(g^{-1}H) = (gg^{-1})H = H.$$

Now, let $x \in gH$. Then $gH = xH$, so we also have

$$(gH)(g^{-1}H) = (xH)(g^{-1}H) = xg^{-1}H.$$

Because coset multiplication is well-defined, $xg^{-1}H = H$. Thus,

$$xg^{-1} = h \in H,$$

so $x = hg \in Hg$.

This shows $gH \subseteq Hg$.

The proof of $Hg \subseteq gH$ is similar.

✓

($\Leftarrow$) Next time.