Recall: A **relation** on a set S is a subset $R \subseteq S \times S$.

The subset R is equivalent to the logical sentence

$$x R y := (x,y) \in R.$$

We usually think of a relation in this way.

An **equivalence relation** on S is a relation $\sim$ such that for all $x, y, z \in S$,

1. $x \sim x$, [reflexive]
2. if $x \sim y$, then $y \sim x$, [symmetric]
3. if $x \sim y$ and $y \sim z$, then $x \sim z$. [transitive]

Examples

- Congruence mod n

Let $n \in \mathbb{N}$. Integers $a, b \in \mathbb{Z}$ are congruent modulo n if

$$n \mid (b-a),$$

i.e., $b-a = nk$ for some $k \in \mathbb{Z}$.
In this case, write $a \equiv b \pmod{n}$.

This is an equivalence relation on $\mathbb{Z}$.
(Math 3345)

- Cardinality

  Two sets A and B have the same cardinality if there exists a bijection $f: A \to B$.

  In this case, write $|A| = |B|$.

  This is an equivalence relation on any set of sets.

- Triangles

  Two triangles are congruent if they have the same 3 side lengths. They are similar if they have the same 3 angles.

  These are both equivalence relations on the set of all triangles in $\mathbb{R}^2$.

- <u>Antiderivatives</u>

  For differentiable functions $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$, write $f \sim g$ if $f' = g'$.

  Then $\sim$ is an equivalence relation on the set of all differentiable functions on $\mathbb{R}$.

- <u>Change of basis</u>

  Let A and B be square $(n \times n)$ matrices with entries in $\mathbb{R}$.

  Write $A \sim B$ if there exists an invertible $n \times n$ matrix P such that

  $$PAP^{-1} = B$$

  Then $\sim$ is an equivalence relation on $M_n(\mathbb{R})$.

Partitions

Def: A <u>partition</u> of a set S is a set of nonempty subsets such that each $x \in S$ is in exactly one of the subsets.

Notation: Let I be an indexing set, and for each $i \in I$ let $X_i \subseteq S$ be a subset.

Then $P = \{X_i\}_{i \in I}$ is a partition of S if

- $X_i \neq \emptyset$ for all $i \in I$
- $X_i \cap X_j = \emptyset$ if $i \neq j$
- $\bigcup_{i \in I} X_i = S$.

Ex: Some partitions of $\mathbb{Z}$:

- A partition into 2 sets

$$X_1 = \{2k \mid k \in \mathbb{Z}\}$$ (evens)

$$X_2 = \{2k+1 \mid k \in \mathbb{Z}\}$$ (odds)

- A partition into 3 sets

$$X_1 = \{3k \mid k \in \mathbb{Z}\}$$

$$X_2 = \{3k+1 \mid k \in \mathbb{Z}\}$$

$$X_3 = \{3k+2 \mid k \in \mathbb{Z}\}$$

- A partition into infinitely many sets

$$X_0 = \{0\},\ X_1 = \{1,-1\},\ X_2 = \{2,-2\}, \ldots$$

<u>Def</u>: Let $\sim$ be an equivalence relation on a set S. For each $x \in S$, define the <u>equivalence class</u> of x to be

$$[x] = \{ y \in S \mid y \sim x \}.$$

<u>Important fact</u>: An equivalence relation on a set S is "the same" as a partition of S. Precisely,

<u>Thm</u>: Let S be a set.

If $\sim$ is an equivalence relation on S, then the equivalence classes partition S.

Conversely, if $P = \{X_i\}_{i \in I}$ is a partition of S, then there is an equivalence relation on X such that $\{X_i\}$ are the equivalence classes.

Proof: Let $x \in S$. By the reflexive property, $x \sim x$. Hence, $x \in [x]$.

This shows that

- every equivalence class is non-empty
- each $x \in S$ is in at least one equivalence class.

We just need to prove that no $x \in S$ can belong to more than one equivalence class.

Suppose, then, that $x \in [y]$. We must prove $[x] = [y]$.

Since $x \in [y]$, we have $x \sim y$. By symmetry, $y \sim x$ also.

Now, let $z \in [y]$. Then $z \sim y$.

By transitivity, $z \sim x$, so $z \in [x]$.
Hence, $[y] \subseteq [x]$.

Similar reasoning shows $[x] \subseteq [y]$,
so $[x] = [y]$.

Thus, the equivalence classes form a partition of S.

✓

Conversely, let $P = \{X_i\}_{i \in I}$ be a partition of S.

Define a relation $\sim$ on S by

$x \sim y \iff$ x and y are in the same subset X_i

Then $\sim$ is

① Reflexive: Each $x \in S$ is in some X_i, so $x \sim x$.

② Symmetric: By definition.

③ Transitive: Suppose $x \sim y$ and $y \sim z$. Then

$$x, y \in X_i \quad \text{and} \quad y, z \in X_j$$

for some i, j.

Since $y \in X_i$ and $y \in X_j$, it must be that $X_i = X_j$. Thus, $x \sim y$.

So $\sim$ is an equivalence relation, with equivalence classes $\{X_i\}_{i \in I}$. ∎