<u>Warm-Up</u>: Check that coset multiplication is well-defined in S_3/A_3.

<u>Thm</u>: Let G be a group and $H \leq G$.

Coset multiplication in G/H is well-defined as

$$(aH)(bH) = abH$$

if and only if $H \trianglelefteq G$.

<u>Proof</u>: ($\Rightarrow$) Last time.

($\Leftarrow$) Assume $H \trianglelefteq G$.

Let $aH, bH \in G/H$ be cosets.

To show $(aH)(bH) = abH$ is well-defined, suppose

$$aH = xH \iff x \in aH$$

and

$$bH = yH. \iff y \in bH$$

Then $x = ah_1$ and $y = bh_2$ for some $h_1, h_2 \in H$.

Now,

$$xy = a\underline{h_1 b}h_2.$$

Since $h_1 b \in Hb = bH$, we have $h_1 b = bh_3$ for some $h_3 \in H$. Then

$$xy = ah_1bh_2 = ab(h_3h_2) \in abH.$$

Thus, $(xH)(yH) = xyH = abH$, so coset multiplication is well-defined. $\blacksquare$

Cor: Let G be a group and $N \trianglelefteq G$ a normal subgroup. Then G/N is a group under coset multiplication, called the <u>quotient group</u> (or <u>factor group</u>) of G by N.

Read G/N as "G modulo N."

Proof: By the theorem, coset multiplication is a binary operation on G/N.

<u>Associativity</u>: follows from associativity in G.

<u>Identity</u>: $eH = H$. "identity coset"

<u>Inverses</u>: $(gH)^{-1} = g^{-1}H$. ∎

Ex: $G = \mathbb{Z}$, $N = \langle 3 \rangle = 3\mathbb{Z}$.

(normal because $\mathbb{Z}$ is abelian)

$$G/N = \mathbb{Z}/3\mathbb{Z} = \{3\mathbb{Z}, 1+3\mathbb{Z}, 2+3\mathbb{Z}\}$$

with coset addition

$$(a+3\mathbb{Z}) + (b+3\mathbb{Z}) = (a+b) + 3\mathbb{Z}.$$

Thus, $\mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}_3$.

Ex: More generally, $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$.

Ex: $G = S_n$, $N = A_n$

(normal because index 2)

Then $S_n/A_n = \{A_n, (12)A_n\}$ is isomorphic to $\mathbb{Z}_2$.

Think about the multiplication

	A_n	$(12)A_n$
A_n	A_n	$(12)A_n$
$(12)A_n$	$(12)A_n$	A_n

as

	even	odd
even	even	odd
odd	odd	even

In general, $G/N \cong \mathbb{Z}_2$ whenever $[G:N] = 2$.

Ex: $G = D_n$, $N = \langle r \rangle$

Think of multiplication in $G/N \cong \mathbb{Z}_2$

	N	sN
N	N	sN
sN	sN	N

as saying

	rotation	reflection
rotation	rotation	reflection
reflection	reflection	rotation