Kernels

Def: Let $\varphi: G \to H$ be a homomorphism of groups. The *kernel* of φ is

$$\ker \varphi = \{ g \in G \mid \varphi(g) = e_H \}$$

That is, $\ker \varphi = \varphi^{-1}(e_H)$ is the "fiber over the identity."

Thm: Let $\varphi: G \to H$ be a group homomorphism. Then $\ker \varphi \trianglelefteq G$.

Proof: Since $\varphi(e_G) = e_H$, we have $e_G \in \ker \varphi$, so $\ker \varphi \neq \emptyset$. If $k_1, k_2 \in \ker \varphi$, then

$$\varphi(k_1 k_2^{-1}) = \varphi(k_1)\varphi(k_2)^{-1} = e_H \cdot e_H^{-1} = e_H,$$

so $k_1 k_2^{-1} \in \ker \varphi$. Thus, $\ker \varphi \leq G$.

Suppose $k \in \ker \varphi$ and $g \in G$.
Then

$$\varphi(gkg^{-1}) = \varphi(g)\underbrace{\varphi(k)}_{e_H}\varphi(g)^{-1}$$
$$= \varphi(g)\varphi(g)^{-1}$$
$$= e_H,$$

proving that $gkg^{-1} \in \ker \varphi$.

That is, $\ker \varphi \trianglelefteq G$.

<u>Ex</u>: $\mathrm{sgn}: S_n \longrightarrow \{1, -1\}$

$$\sigma \longmapsto \begin{cases} 1 & \sigma \text{ even} \\ -1 & \sigma \text{ odd} \end{cases}$$

has $\ker \mathrm{sgn} = A_n$.

<u>Ex</u>: $\det: GL_n(\mathbb{R}) \longrightarrow \mathbb{R}^\times$

has $\ker \det = SL_n(\mathbb{R})$.

Ex: Let G be any group, and $g \in G$.
There is a homomorphism

$$\varphi : \mathbb{Z} \to G$$
$$n \mapsto g^n.$$

Then $\ker \varphi = \{0\}$ if $|g| = \infty$ and
$\ker \varphi = n\mathbb{Z}$ if $|g| = n$.

We've proved that kernels are normal.
Conversely, any normal subgroup is a kernel:

Thm: Let G be a group and $N \trianglelefteq G$.
Then

$$\pi : G \to G/N$$
$$g \mapsto gN$$

is a group homomorphism, and
$\ker \pi = N$.

<u>Note</u>: π is called the <u>natural</u> or <u>canonical</u> homomorphism of the quotient group G/N.

<u>Proof</u>: Let $g_1, g_2 \in G$. Then

$$\pi(g_1 g_2) = g_1 g_2 N = (g_1 N)(g_2 N)$$
$$= \pi(g_1)\,\pi(g_2),$$

so π is a homomorphism.

Now, $g \in \ker \pi \iff \pi(g) = N$

$\iff gN = N$

$\iff g \in N.$

$\blacksquare$

Thm (First Isomorphism Theorem)

Think: "Fundamental Theorem of Homomorphisms"

Let $\varphi: G \to H$ be a group homomorphism. Then

- $\ker\varphi \trianglelefteq G$ ✓
- $\varphi(G) \leq H$ ✓
- $G/\ker\varphi \cong \varphi(G)$

More precisely, there is a unique homomorphism

$$\eta: G/\ker\varphi \longrightarrow H$$

such that $\eta \circ \pi = \varphi$, where $\pi: G \to G/\ker\varphi$ is the natural projection.

Then η is an isomorphism onto $\varphi(G)$.

Picture:

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[dr, "\pi"'] & \varphi(G) \leq H \\ & G/\ker\varphi \arrow[u, dashed, "\eta"'] \end{tikzcd}$$