Thm (First Isomorphism Theorem)

Think: "Fundamental Theorem of Homomorphisms"

Let $\varphi : G \to H$ be a group homomorphism. Then

- $\ker \varphi \trianglelefteq G$ ✓
- $\varphi(G) \leq H$ ✓
- $G/\ker \varphi \cong \varphi(G)$

More precisely, there is a unique homomorphism

$$\eta : G/\ker \varphi \longrightarrow H$$

such that $\eta \circ \pi = \varphi$, where $\pi : G \to G/\ker \varphi$ is the natural projection.

Then η is an isomorphism onto $\varphi(G)$.

Picture:

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \varphi(G) \leq H \\ G/\ker\varphi \arrow[ur, dashed, "\eta"'] & \end{tikzcd}$$

Note: The condition $\eta \circ \pi = \varphi$ forces

$$\eta(gK) = (\eta \circ \pi)(g) = \varphi(g) \qquad (*)$$

for every coset $gK \in G/K$ $(K = \ker\varphi)$.

So uniqueness is automatic – we just need to check $(*)$ is well-defined.

Proof: Let $K = \ker\varphi$.

We begin by proving that for any $g_1, g_2 \in G$,

$$g_1K = g_2K \iff \varphi(g_1) = \varphi(g_2). \qquad (\star)$$

($\Rightarrow$) Suppose $g_1 K = g_2 K$. Then $g_1 = g_2 k$ for some $k \in K$. Thus,

$$\varphi(g_1) = \varphi(g_2 k) = \varphi(g_2)\underbrace{\cancel{\varphi(k)}}_{=e}$$
$$= \varphi(g_2).$$

✓

($\Leftarrow$) Suppose $\varphi(g_1) = \varphi(g_2)$. Then

$$e = \varphi(g_1)^{-1}\varphi(g_2) = \varphi(g_1^{-1}g_2),$$

so $g_1^{-1}g_2 \in K$. Thus, $g_1 K = g_2 K$.

✓

Now, we can define

$$\eta : G/K \longrightarrow H$$
$$gK \longmapsto \varphi(g).$$

By (☆), we know that η is well-defined ($\Rightarrow$ direction) and η is injective ($\Leftarrow$ direction).

Thus, η is a bijection onto its image, which by construction is $\varphi(G)$.

It only remains to prove φ is a homomorphism. We have

$$\begin{aligned}
\eta((g_1K)(g_2K)) &= \eta(g_1g_2K) \\
&= \varphi(g_1g_2) \\
&= \varphi(g_1)\,\varphi(g_2) \\
&= \eta(g_1K)\,\eta(g_2K).
\end{aligned}$$

$\blacksquare$

Corollaries

Cor 1: If $\varphi: G \to H$ is a surjective group homomorphism, then

$$G/\ker\varphi \cong H.$$

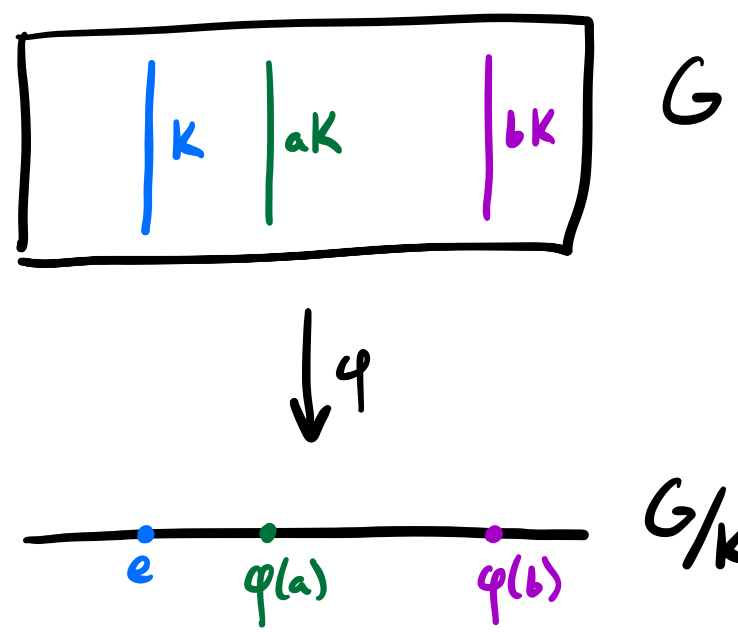
Proof: Surjective means $\varphi(G) = H$. $\blacksquare$

Cor 2: Let $\varphi: G \to H$ be a group homomorphism. Then, for $g_1, g_2 \in G$, $\varphi(g_1) = \varphi(g_2)$ if and only if $g_1K = g_2K$, where $K = \ker\varphi$.

That is, the fibers of φ are precisely the cosets of the kernel.

Proof: This is just a restatement of $(\star)$. $\blacksquare$

Picture of Cor 2



$G/K \cong \varphi(G) \leq H$

Cor 3: If $\varphi: G \to H$ is a group homomorphism, then φ is injective if and only if $\ker \varphi = \{e\}$.

Proof: $\varphi(g_1) = \varphi(g_2) \iff g_1 K = g_2 K$
$\iff g_2^{-1} g_1 \in K = \ker \varphi$

$\blacksquare$

Ex: Consider $\varphi : \mathbb{R}^2 \to \mathbb{R}$.
$$(x,y) \mapsto x+y$$

Then φ is a surjective homomorphism (Check this!), so

$$\mathbb{R}^2 / \ker\varphi \cong \mathbb{R}.$$

What does this look like?

$$K = \ker\varphi = \{ (x,y) \in \mathbb{R}^2 \mid \underbrace{\varphi(x,y)}_{= x+y} = 0 \}$$

is just the line $y = -x$ in $\mathbb{R}^2$.

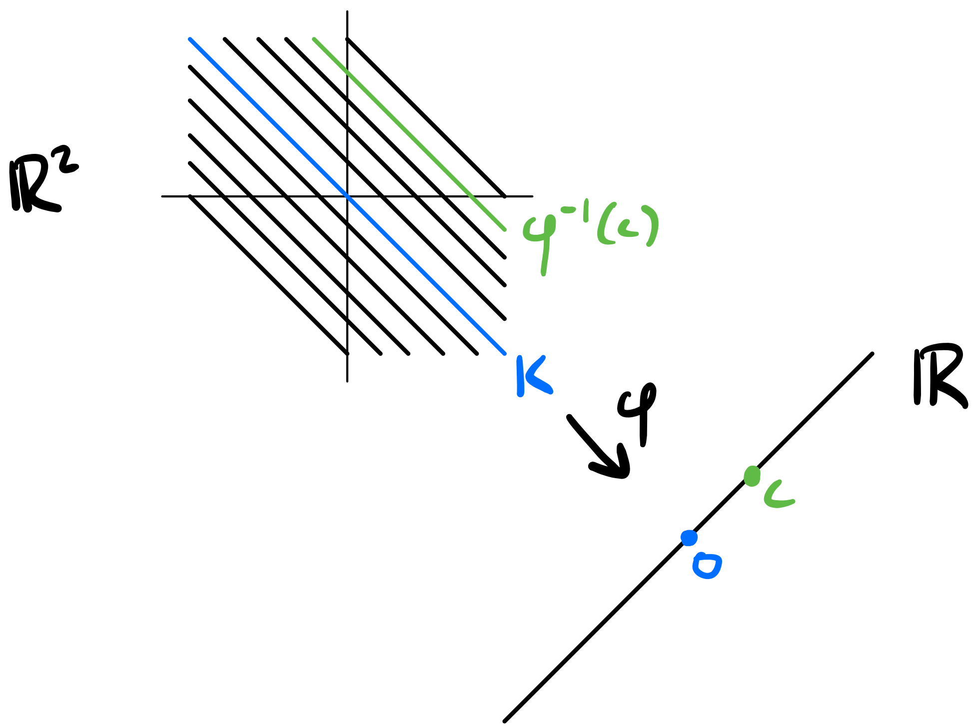
Given any $c \in \mathbb{R}$, the fiber over c is

$$\varphi^{-1}(c) = \{ (x,y) \in \mathbb{R}^2 \mid \underbrace{\varphi(x,y)}_{= x+y} = c \}.$$

That is, $\varphi^{-1}(c)$ is the line $y=-x+c$. This is a coset (translate) of K:

$$\varphi^{-1}(c) = (c,0) + K.$$

So $\mathbb{R}^2/K$ is the set of lines in $\mathbb{R}^2$ with slope -1.



The First Isomorphism Theorem tells us that these lines form a group isomorphic to $\mathbb{R}$.