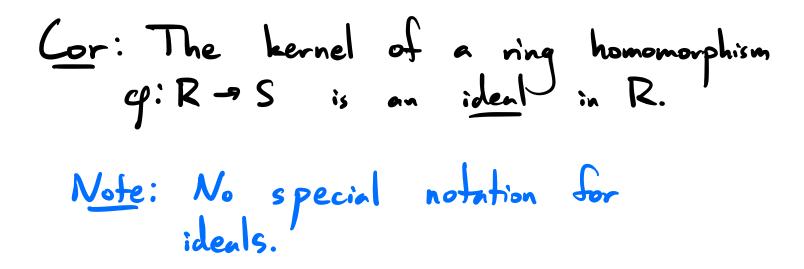
$$\frac{\text{Warm-Up}: \text{ Define}}{\text{cp}: \mathbb{R}[x] \longrightarrow \mathbb{R}}$$
$$p(x) \longmapsto p(5)$$

Show op is a ring hom.
Find the image and kernel of cp.

Proof: We already know
$$\varphi(R)$$
 is a
subgroup of S and ker φ is a subgroup
of R .
To show they are subrings, we just need
closure under multiplication.
(i) Let $s_{i1} s_{i2} \in \varphi(R)$. Then
 $s_{i} = \varphi(r_{i})$ and $s_{i2} = \varphi(r_{i2})$
for some $r_{i3}r_{i2} \in R$.
Now,
 $s_{i}s_{i2} = \varphi(r_{i}) \varphi(r_{i2}) = \varphi(r_{i}r_{i2}) \in \varphi(R)$.
(i) Let $a_{i3}, a_{i2} \in \ker \varphi$.
Now,
 $s_{i}s_{i2} = \varphi(a_{i}) \cdot \varphi(a_{i3}) = 0.0 = 0$,
so $a_{i}a_{i2} \in \ker \varphi$.

(3) Now, let
$$a \in ker q$$
 and $r \in R$.
Similar to (2), we have
 $q(ra) = q(r) q(a) = q(r) \cdot 0 = 0$
and
 $q(ar) = q(a) q(r) = 0 \cdot q(r) = 0$.
So ra, ar ϵ ker q .



Ex: Let
$$K$$
 be a commutative ring with I .
For any $a \in R$, define
 $(a) = \{ra \mid r \in R\}$.
Then (a) is an ideal, called the
principal ideal generated by a .

Proof: We show (a) is an additive
subgroup:
Identity:
$$O = O \cdot a \in (a)$$
.
Closure: $r_1 a + r_2 a = (r_1 + r_2)a \in (a) /$
Inverses: $-(r_1 a) = (-r_1)a \in (a) /$

Now, for any
$$ra \in (a)$$
 and $s \in R$,
we have
 $S(ra) = (sr)a \in (a)$,
So (a) is an ideal

Thm: Every ideal in Z is principal.
Proof: Certainly
$$\{0\} = (0)$$
 is principal.
Let $I \neq \{0\}$ be an ideal. Then
I contains some positive integer (using?)
so by Well-Ordering it contains a beast
positive integer $n \in I$.
For any $a \in I$, the division algorithm yields
 $a = ng + r$,
where $0 \leq r < n$. But $r = a - ng \in I$,
so $r = 0$ by minimality of n . $i = 1$.
Thus, $a = ng$, so $I = (n) = nZ$.

Ex: Similarly, every ideal in
$$R[x]$$

is principal.
Why? Polynomial long division!
Ex: The set
 $I = \{p(x, y) \mid p(0, 0) = 0\} \in R[x, y]$
is an ideal which is not principal.
Why? Both $x \in I$ and $y \in I$,
but there is no polynomial
 $g(x, y)$ such that both x and y
are multiples of g .

Def: Let R be a ring with I.
The characteristic of R is
the smallest positive integer
such that

$$n \cdot l = \underbrace{l + l + \dots + l}_{n \text{ times}} = 0.$$

If there is no such positive
integer, then we say R has
characteristic O.
Ex: Z_n has characteristic n
Ex: Z, R, R, C, R[x], ... have characteristic O.

Alternative perspective:
There is a ring homomorphism

$$\varphi: \mathbb{Z} \rightarrow \mathbb{R}$$

defined by
 $\int \frac{1+1+\dots+1}{k} k > 0$

$$q(k) = k \cdot l = 2 0 k = 0$$

 $(-1) + (-1) + (-1) k < 0$
 $k = 0$
 $k = 0$

Then her
$$cp = n\mathbb{Z}$$
 is an ideal
of \mathbb{Z} , and n is the characteristic
of \mathbb{R} .