Thm: If R is an integral domain, then the characteristic of R is prime or 0.

Proof: Suppose the characteristic of R is $n > 0$.

If $n = k\ell$, $k, \ell \in \mathbb{N}$, then

$$0 = n \cdot 1 = (k \cdot 1)(\ell \cdot 1).$$

Since R is an integral domain, either $k \cdot 1 = 0$ or $\ell \cdot 1 = 0$. Say $k \cdot 1 = 0$.

But $k \leq n$, so by minimality of n, we have $k = n$ and $\ell = 1$.

Thus, n is prime. $\blacksquare$

If $I \subseteq R$ is a <u>subring</u>, then in particular it is an additive (normal) subgroup. So

$$R/I = \{r + I \mid r \in R\}$$

is an abelian group.

When is it a ring?

<u>Thm</u>: Let $I \subseteq R$ be a subring. Then multiplication of cosets

$$(a+I)(b+I) = ab + I$$

is well-defined if and only if I is an ideal.

Proof: ($\Rightarrow$) Suppose coset multiplication is well-defined. Then for any $r \in R$, $a \in I$ we have

$$(r+I)(a+I) = ra + I.$$

But $a + I = I = 0 + I$, so

$$\begin{aligned}(r+I)(a+I) &= (r+I)(0+I) \\ &= r \cdot 0 + I \\ &= 0 + I \\ &= I.\end{aligned}$$

Thus, $ra + I = I$, so $ra \in I$.

Similarly, $ar \in I$.

So I is an ideal.

($\Leftarrow$) Conversely, suppose I is an ideal. Suppose

$$a+I = x+I \quad \text{and} \quad b+I = y+I.$$

We need to show $ab+I = xy+I$.

Well, $x = a+i$ and $y = b+j$ for some $i, j \in I$. Then

$$xy = (a+i)(b+j) = ab + \underbrace{aj + ib + ij}_{\in I}$$

so $xy \in ab+I$, i.e. $xy+I = ab+I$.

$\blacksquare$

Cor: Let R be a ring and $I \subseteq R$ an ideal. Then R/I is a ring under coset addition and multiplication:

$$(a+I) + (b+I) = (a+b) + I$$

$$(a+I)\cdot(b+I) = ab + I.$$

Proof: Since the operations in R/I are the operations of R applied to coset representatives, R/I is a ring because R is a ring. $\blacksquare$

We have already seen that kernels of ring homomorphisms are ideals.

We now see that every ideal is a kernel.

Cor: Let R be a ring and $I \trianglelefteq R$ an ideal. The canonical projection

$$\pi : R \to R/I$$
$$a \mapsto a + I$$

is a ring homomorphism with $\ker \pi = I$