Thm (First Isomorphism Theorem for Rings)

Let $\varphi: R \to S$ be a ring homomorphism. Then

- $\ker \varphi \subseteq R$ is an ideal.
- $\varphi(R) \subseteq S$ is a subring.
- $R/\ker \varphi \cong \varphi(R)$.

More precisely, there is a unique homomorphism

$$\eta: R/\ker \varphi \to S$$

such that $\eta \circ \pi = \varphi$, where $\pi: R \to R/\ker \varphi$ is the natural projection.

Then η is an isomorphism onto $\varphi(R)$.

Proof: Let $K = \ker \varphi$.

We have already proved that K is an ideal in R and $\varphi(R)$ is a subring of S.

By the First Isomorphism Theorem for groups, we know that

$$\eta: R/K \to S$$

$$a + K \mapsto \varphi(a)$$

is a well-defined injective group homomorphism with image $\varphi(R)$.

It remains to show η is a ring homomorphism, i.e., it respects multiplication.

We have

$$\begin{aligned}\eta\big((a+K)(b+K)\big) &= \eta(ab+K)\\ &= \varphi(ab)\\ &= \varphi(a)\varphi(b)\\ &= \eta(a+K)\,\eta(b+K).\end{aligned}$$

$\boxtimes$

Ex: Consider

$$\begin{aligned}ev_0 : \mathbb{Z}[x] &\to \mathbb{Z}\\ p(x) &\mapsto p(0).\end{aligned}$$

Then

$$\begin{aligned}\ker ev_0 &= \{p(x) \mid p(0)=0\}\\ &= \{x\cdot q(x) \mid q(x)\in\mathbb{Z}[x]\\ &= (x),\end{aligned}$$

and ev_0 is surjective.

Thus,

$$\mathbb{Z}[x]/(x) \cong \mathbb{Z}.$$

Think: Set $x=0$. Then two polynomials $p(x)$ and $q(x)$ are identified if $p(0)=q(0)$.

More precisely, let $K_1 = (x)$. Then

$$p(x) + K = q(x) + K$$

$$\iff p(0) = q(0).$$

e.g. $(3+5x+8x^3)+K = (3+2x^2)+K$,

since

$$(3+5x+8x^3)-(3+2x^2) = 5x - 2x^2 + 8x^3 \in K.$$

$= x(5-2x+8x^2)$

Ex: How is $\mathbb{Z}[x]/(x^2)$ different?

(Note: (x^2) not the kernel of a familiar homomorphism.)

Think: Set $x^2=0$, but $x \neq 0$.
So $p(x)$ and $q(x)$ get identified if $p(0)=q(0)$ and $p'(0)=q'(0)$.

Let $K_2 = (x^2)$. Then, e.g.,

$$(1+5x-4x^2)+K_2 = (1+5x+x^9)+K_2$$

since

$$(1+5x-4x^2)-(1+5x+x^9) = -4x^2-x^9 \in K_2.$$

$$= x^2(-4-x^7)$$

Observe that $\mathbb{Z}[x]/(x^2)$ has zero divisors:

$$(4x + K_2)(3x + K_2) = 12x^2 + K_2$$
$$= K_2,$$

but $4x + K_2,\ 3x + K_2 \neq K_2$

Remember, $K_2 = 0 + K_2$ is "0" in R/K_2.

<u>Def</u>: An ideal $P \subseteq R$ is <u>prime</u> if for all $a, b \in R$,

$$ab \in R \Rightarrow a \in R \text{ or } b \in R.$$

<u>Ex</u>: In $\mathbb{Z}[x]$, (x) is prime but (x^2) is not.