In many important physical problems there are two or more independent variables, so the corresponding mathematical models involve partial, rather than ordinary, differential equations. This chapter treats one important method for solving partial differential equations, a method known as separation of variables. Its essential feature is the replacement of the partial differential equation by a set of ordinary differential equations, which must be solved subject to given initial or boundary conditions. The first section of this chapter deals with some basic properties of boundary value problems for ordinary differential equations. The desired solution of the partial differential equation is then expressed as a sum, usually an infinite series, formed from solutions of the ordinary differential equations. In many cases we ultimately need to deal with a series of sines and/or cosines, so part of the chapter is devoted to a discussion of such series, which are known as Fourier series. With the necessary mathematical background in place, we then illustrate the use of separation of variables in a variety of problems arising from heat conduction, wave propagation, and potential theory.
10.1 Two-Point Boundary Value Problems

Up to this point in the book we have dealt with initial value problems, consisting of a differential equation together with suitable initial conditions at a given point. A typical example, which was discussed at length in Chapter 3, is the differential equation

\[ y'' + p(t)y' + q(t)y = g(t), \]  

with the initial conditions

\[ y(t_0) = y_0, \quad y'(t_0) = y'_0. \]  

Physical applications often lead to another type of problem, one in which the value of the dependent variable \( y \) or its derivative is specified at two different points. Such conditions are called boundary conditions to distinguish them from initial conditions that specify the value of \( y \) and \( y' \) at the same point. A differential equation and suitable boundary conditions form a two-point boundary value problem. A typical example is the differential equation

\[ y'' + p(x)y' + q(x)y = g(x) \]  

with the boundary conditions

\[ y(\alpha) = y_0, \quad y(\beta) = y_1. \]  

The natural occurrence of boundary value problems usually involves a space coordinate as the independent variable, so we have used \( x \) rather than \( t \) in Eqs. (3) and (4). To solve the boundary value problem (3), (4) we need to find a function \( y = \phi(x) \) that satisfies the differential equation (3) in the interval \( \alpha < x < \beta \) and that takes on the specified values \( y_0 \) and \( y_1 \) at the endpoints of the interval. Usually, we first seek the general solution of the differential equation and then use the boundary conditions to determine the values of the arbitrary constants.

Boundary value problems can also be posed for nonlinear differential equations, but we will restrict ourselves to a consideration of linear equations only. An important classification of linear boundary value problems is whether they are homogeneous or nonhomogeneous. If the function \( g \) has the value zero for each \( x \), and if the boundary values \( y_0 \) and \( y_1 \) are also zero, then the problem (3), (4) is called homogeneous. Otherwise, the problem is nonhomogeneous.
Although the initial value problem (1), (2) and the boundary value problem (3), (4) may superficially appear to be quite similar, their solutions differ in some very important ways. Under mild conditions on the coefficients initial value problems are certain to have a unique solution. On the other hand, boundary value problems under similar conditions may have a unique solution, but they may also have no solution or, in some cases, infinitely many solutions. In this respect, linear boundary value problems resemble systems of linear algebraic equations.

Let us recall some facts (see Section 7.3) about the system

\[ Ax = b, \quad (5) \]

where \( A \) is a given \( n \times n \) matrix, \( b \) is a given \( n \times 1 \) vector, and \( x \) is an \( n \times 1 \) vector to be determined. If \( A \) is nonsingular, then the system (5) has a unique solution for any \( b \). However, if \( A \) is singular, then the system (5) has no solution unless \( b \) satisfies a certain additional condition, in which case the system has infinitely many solutions. Now consider the corresponding homogeneous system

\[ Ax = 0, \quad (6) \]

obtained from the system (5) when \( b = 0 \). The homogeneous system (6) always has the solution \( x = 0 \). If \( A \) is nonsingular, then this is the only solution, but if \( A \) is singular, then there are infinitely many (nonzero) solutions. Note that it is impossible for the homogeneous system to have no solution. These results can also be stated in the following way: The nonhomogeneous system (5) has a unique solution if and only if the homogeneous system (6) has only the solution \( x = 0 \), and the nonhomogeneous system (5) has either no solution or infinitely many if and only if the homogeneous system (6) has nonzero solutions.

We now turn to some examples of linear boundary value problems that illustrate very similar behavior. A more general discussion of linear boundary value problems appears in Chapter 11.

**Example 1**

Solve the boundary value problem

\[ y'' + 2y = 0, \quad y(0) = 1, \quad y(\pi) = 0. \quad (7) \]

The general solution of the differential equation (7) is

\[ y = c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x. \quad (8) \]

The first boundary condition requires that \( c_1 = 1 \). The second boundary condition implies that \( c_1 \cos \sqrt{2}\pi + c_2 \sin \sqrt{2}\pi = 0 \), so \( c_1 \cos \sqrt{2}\pi + c_2 \sin \sqrt{2}\pi = 0 \), so \( c_2 = -\cot \sqrt{2}\pi \approx -0.2762 \). Thus the solution of the boundary value problem (7) is

\[ y = \cos \sqrt{2}x - \cot \sqrt{2}\pi \sin \sqrt{2}x. \quad (9) \]
This example illustrates the case of a nonhomogeneous boundary value problem with a unique solution.

**Example 2**

Solve the boundary value problem

\[
y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = a,
\]

where \( a \) is a given number.

The general solution of this differential equation is

\[
y = c_1 \cos x + c_2 \sin x,
\]

and from the first boundary condition we find that \( c_1 = 1 \). The second boundary condition now requires that \( -c_1 = a \). These two conditions on \( c_1 \) are incompatible if \( a \neq -1 \), so the problem has no solution in that case. However, if \( a = -1 \), then both boundary conditions are satisfied provided that \( c_1 = 1 \), regardless of the value of \( c_2 \). In this case there are infinitely many solutions of the form

\[
y = \cos x + c_2 \sin x,
\]

where \( c_2 \) remains arbitrary. This example illustrates that a nonhomogeneous boundary value problem may have no solution, and also that under special circumstances it may have infinitely many solutions.

Corresponding to the nonhomogeneous boundary value problem (3), (4) is the homogeneous problem consisting of the differential equation

\[
y'' + p(x)y' + q(x)y = 0
\]

and the boundary conditions

\[
y(\alpha) = 0, \quad y(\beta) = 0.
\]

Observe that this problem has the solution \( y = 0 \) for all \( x \), regardless of the coefficients \( p(x) \) and \( q(x) \). This solution is often called the trivial solution and is rarely of interest. What we usually want to know is whether the problem has other, nonzero solutions. Consider the following two examples.
**Example 3**

Solve the boundary value problem

\[ y'' + 2y = 0, \quad y(0) = 0 \quad y(\pi) = 0. \tag{15} \]

The general solution of the differential equation is again given by Eq. (8),

\[ y = c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x. \]

The first boundary condition requires that \( c_1 = 0 \) and the second boundary condition leads to \( c_2 \sin \sqrt{2} \pi = 0 \). Since \( \sin \sqrt{2} \pi \neq 0 \), it follows that \( c_2 = 0 \) also. Consequently, \( y = 0 \) for all \( x \) is the only solution of the problem (15). This example illustrates that a homogeneous boundary value problem may have only the trivial solution \( y = 0 \).

**Example 4**

Solve the boundary value problem

\[ y'' + y = 0, \quad y(0) = 0 \quad y(\pi) = 0. \tag{16} \]

The general solution is given by Eq. (11),

\[ y = c_1 \cos x + c_2 \sin x, \]

and the first boundary condition requires that \( c_1 = 0 \). Since \( \sin \pi = 0 \), the second boundary condition is also satisfied regardless of the value of \( c_2 \). Thus the solution of the problem (16) is \( y = c_2 \sin x \), where \( c_2 \) remains arbitrary. This example illustrates that a homogeneous boundary value problem may have infinitely many solutions.

Examples 1 through 4 illustrate (but of course do not prove) that there is the same relationship between homogeneous and nonhomogeneous linear boundary value problems as there is between homogeneous and nonhomogeneous linear algebraic systems. A nonhomogeneous boundary value problem (Example 1) has a unique solution, and the corresponding homogeneous problem (Example 3) has only the trivial solution. Further, a nonhomogeneous problem (Example 2) has either no solution or infinitely many, and the corresponding homogeneous problem (Example 4) has nontrivial solutions.

**Eigenvalue Problems.** Recall the matrix equation

\[ Ax = \lambda x \tag{17} \]
that we discussed in Section 7.3. Equation (17) has the solution $\mathbf{x} = 0$ for every value of $\lambda$, but for certain values of $\lambda$, called eigenvalues, there are also nonzero solutions, called eigenvectors. The situation is similar for boundary value problems.

Consider the problem consisting of the differential equation

$$y'' + \lambda y = 0,$$  \hfill (18)

together with the boundary conditions

$$y(0) = 0, \quad y(\pi) = 0.$$  \hfill (19)

Observe that the problem (18), (19) is the same as the problems in Examples 3 and 4 if $\lambda = 2$ and $\lambda = 1$, respectively. Recalling the results of these examples, we note that for $\lambda = 2$, Eqs. (18), (19) have only the trivial solution $y = 0$, while for $\lambda = 1$, the problem (18), (19) has other, nontrivial solutions. By extension of the terminology associated with Eq. (17), the values of $\lambda$ for which nontrivial solutions of (18), (19) occur are called eigenvalues, and the nontrivial solutions themselves are called eigenfunctions. Restating the results of Examples 3 and 4, we have found that $\lambda = 1$ is an eigenvalue of the problem (18), (19) and that $\lambda = 2$ is not. Further, any nonzero multiple of $\sin x$ is an eigenfunction corresponding to the eigenvalue $\lambda = 1$.

Let us now turn to the problem of finding other eigenvalues and eigenfunctions of the problem (18), (19). We need to consider separately the cases $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$, since the form of the solution of Eq. (18) is different in each of these cases. Suppose first that $\lambda = 0$. To avoid the frequent appearance of radical signs, it is convenient to let $\mu = \frac{1}{\sqrt{\lambda}}$ and to rewrite Eq. (18) as

$$y'' + \mu^2 y = 0.$$  \hfill (20)

The characteristic polynomial equation for Eq. (20) is $r^2 + \mu^2 = 0$ with roots $r = \pm i\mu$, so the general solution is

$$y = c_1 \cos \mu x + c_2 \sin \mu x.$$  \hfill (21)

Note that $\mu$ is nonzero (since $\lambda = 0$) and there is no loss of generality if we also assume that $\mu$ is positive. The first boundary condition requires that $c_1 = 0$, and then the second boundary condition reduces to

$$c_2 \sin \mu \pi = 0.$$  \hfill (22)

We are seeking nontrivial solutions so we must require that $c_2 \neq 0$. Consequently, $\sin \mu \pi$ must be zero, and our task is to choose $\mu$ so that this will occur. We know that the sine function has the value zero at every integer multiple of $\pi$, so we can choose $\mu$ to be any (positive) integer. The corresponding values of $\lambda$ are the squares of the positive integers, so we have determined that
\[ \lambda_1 = 1, \quad \lambda_2 = 4, \quad \lambda_3 = 9, \quad \ldots, \quad \lambda_n = n^2, \quad \ldots \tag{23} \]

are eigenvalues of the problem (18), (19). The eigenfunctions are given by Eq. (21) with \( c_1 = 0 \), so they are just multiples of the functions \( \sin nx \) for \( n = 1, 2, 3, \ldots \). Observe that the constant \( c_2 \) in Eq. (21) is never determined, so eigenfunctions are determined only up to an arbitrary multiplicative constant [just as are the eigenvectors of the matrix problem (17)]. We will usually choose the multiplicative constant to be 1 and write the eigenfunctions as

\[ y_1(x) = \sin x, \quad y_2(x) = \sin 2x, \quad \ldots, \quad y_n(x) = \sin nx, \quad \ldots \tag{24} \]

remembering that multiples of these functions are also eigenfunctions.

Now let us suppose that \( \lambda < 0 \). If we let \( \lambda = -\mu^2 \), then Eq. (18) becomes

\[ y'' - \mu^2 y = 0. \tag{25} \]

The characteristic equation for Eq. (25) is \( r^2 - \mu^2 = 0 \) with roots \( r = \pm \mu \), so its general solution can be written as

\[ y = c_1 \cosh \mu x + c_2 \sinh \mu x. \tag{26} \]

We have chosen the hyperbolic functions \( \cosh \mu x \) and \( \sinh \mu x \), rather than the exponential functions \( \exp(\mu x) \) and \( \exp(-\mu x) \), as a fundamental set of solutions for convenience in applying the boundary conditions. The first boundary condition requires that \( c_1 = 0 \), and then the second boundary condition gives \( c_2 \sinh \mu \pi = 0 \). Since \( \mu \neq 0 \), it follows that \( \sinh \mu \pi \neq 0 \), and therefore we must have \( c_2 = 0 \). Consequently, \( y = 0 \) and there are no nontrivial solutions for \( \lambda < 0 \). In other words, the problem (18), (19) has no negative eigenvalues.

Finally, consider the possibility that \( \lambda = 0 \). Then Eq. (18) becomes

\[ y'' = 0, \tag{27} \]

and its general solution is

\[ y = c_1 x + c_2. \tag{28} \]

The boundary conditions (19) can be satisfied only by choosing \( c_1 = 0 \) and \( c_2 = 0 \), so there is only the trivial solution \( y = 0 \) in this case as well. That is, \( \lambda = 0 \) is not an eigenvalue.

To summarize our results: We have shown that the problem (18), (19) has an infinite sequence of positive eigenvalues \( \lambda_n = n^2 \) for \( n = 1, 2, 3, \ldots \) and that the corresponding eigenfunctions are
proportional to $\sin n\pi x$. Further, there are no other real eigenvalues. There remains the possibility that there might be some complex eigenvalues; recall that a matrix with real elements may very well have complex eigenvalues. In Problem 23 we outline an argument showing that the particular problem (18), (19) cannot have complex eigenvalues. Later, in Section 11.2, we discuss an important class of boundary value problems that includes (18), (19). One of the useful properties of this class of problems is that all their eigenvalues are real.

In later sections of this chapter we will often encounter the problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0,$$  

(29)

which differs from the problem (18), (19) only in that the second boundary condition is imposed at an arbitrary point $x = L$ rather than at $x = \pi$. The solution process for $\lambda > 0$ exactly the same as before up to the step where the second boundary condition is applied. For the problem (29) this condition requires that

$$c_2 \sin \mu L = 0$$  

(30)

rather than Eq. (22), as in the former case. Consequently, $\mu L$, must be an integer multiple of $\pi$, so $\mu = n\pi/L$, where $n$ is a positive integer. Hence the eigenvalues and eigenfunctions of the problem (29) are given by

$$\lambda_n = n^2 \pi^2 / L^2, \quad y_n(x) = \sin(n\pi x/L), \quad n = 1, 2, 3, \ldots$$  

(31)

As usual, the eigenfunctions $y_n(x)$ are determined only up to an arbitrary multiplicative constant. In the same way as for the problem (18), (19), you can show that the problem (29) has no eigenvalues or eigenfunctions other than those in Eq. (31).

The problems following this section explore to some extent the effect of different boundary conditions on the eigenvalues and eigenfunctions. A more systematic discussion of two-point boundary and eigenvalue problems appears in Chapter 11.

**PROBLEMS**

In each of Problems 1 through 13 either solve the given boundary value problem or else show that it has no solution.

1. $y'' + y = 0, \quad y(0) = 0, \quad y'(\pi) = 1$

2. $y'' + 2y = 0, \quad y'(0) = 1, \quad y'(\pi) = 0$
3. $y'' + y = 0$,   $y(0) = 0$,   $y(L) = 0$

4. $y'' + y = 0$,   $y'(0) = 1$,   $y(L) = 0$

5. $y'' + y = x$,   $y(0) = 0$,   $y(\pi) = 0$

6. $y'' + 2y = x$,   $y(0) = 0$,   $y(\pi) = 0$

7. $y'' + 4y = \cos x$,   $y(0) = 0$,   $y(\pi) = 0$

8. $y'' + 4y = \sin x$,   $y(0) = 0$,   $y(\pi) = 0$

9. $y'' + 4y = \cos x$,   $y'(0) = 0$,   $y'(\pi) = 0$

10. $y'' + 3y = \cos x$,   $y'(0) = 0$,   $y'(\pi) = 0$

11. $x^2y'' - 2xy' + 2y = 0$,   $y(1) = -1$,   $y(2) = 1$

12. $x^2y'' + 3xy' + y = x^2$,   $y(1) = 0$,   $y(e) = 0$

13. $x^2y'' + 5xy' + (4 + \pi^2)y = \ln x$,   $y(1) = 0$,   $y(e) = 0$
In each of Problems 14 through 20 find the eigenvalues and eigenfunctions of the given boundary value problem. Assume that all eigenvalues are real.

14. \( y'' + \lambda y = 0, \ y(0) = 0, \ y'(\pi) = 0 \)

15. \( y'' + \lambda y = 0, \ y'(0) = 0, \ y(\pi) = 0 \)

16. \( y'' + \lambda y = 0, \ y'(0) = 0, \ y'(\pi) = 0 \)

17. \( y'' + \lambda y = 0, \ y'(0) = 0, \ y(L) = 0 \)

18. \( y'' + \lambda y = 0, \ y'(0) = 0, \ y'(L) = 0 \)

19. \( y'' - \lambda y = 0, \ y(0) = 0, \ y'(L) = 0 \)

20. \( x^2 y'' - xy' + \lambda y = 0, \ y(1) = 0, \ y(L) = 0, \ L > 1 \)

21. The axially symmetric laminar flow of a viscous incompressible fluid through a long straight tube of circular cross section under a constant axial pressure gradient is known as Poiseuille flow. The axial velocity \( w \) is a function of the radial variable \( r \) only and satisfies the boundary value problem

\[
w'' + \frac{1}{r} w' = -\frac{G}{\mu}, \quad w(R) = 0, \quad w(r) \text{ bounded for } 0 < r < R,
\]

where \( R \) is the radius of the tube, \( G \) is the pressure gradient, and \( \mu \) is the coefficient of viscosity of the fluid.

(a) Find the velocity profile \( w(r) \).

(b) By integrating \( w(r) \) over a cross section, show that the total flow rate \( Q \) is given by
\[ Q = \pi R^4 G / 8 \mu. \]

Since \( Q, R, \) and \( G \) can be measured, this result provides a practical way to determine the viscosity \( \mu. \)

(c) Suppose that \( R \) is reduced to \( 3/4 \) of its original value. What is the corresponding reduction in \( Q \)? This result has implications for blood flow through arteries constricted by plaque.

22. Consider a horizontal metal beam of length \( L \) subject to a vertical load \( f(x) \) per unit length. The resulting vertical displacement in the beam \( y(x) \) satisfies the differential equation

\[ EI \frac{d^4 y}{dx^4} = f(x), \]

where \( E \) is Young's modulus and \( I \) is the moment of inertia of the cross section about an axis through the centroid perpendicular to the \( xy \)-plane. Suppose that \( f(x)/EI \) is a constant \( k \). For each of the boundary conditions given below solve for the displacement \( y(x) \), and plot \( y \) versus \( x \) in the case that \( L = 1 \) and \( k = -1 \).

(a) Simply supported at both ends: \( y(0) = y''(0) = y(L) = y''(L) = 0 \)

(b) Clamped at both ends: \( y(0) = y'(0) = y(L) = y'(L) = 0 \)

(c) Clamped at \( x = 0 \), free at \( x = L \): \( y(0) = y'(0) = y''(L) = y'''(L) = 0 \)

23. In this problem we outline a proof that the eigenvalues of the boundary value problem (18), (19) are real.

(a) Write the solution of Eq. (18) as \( y = k_1 \exp(i \mu x) + k_2 \exp(-i \mu x), \) where \( \lambda = u^2 \), and impose the boundary conditions (19). Show that nontrivial solutions exist if and only if

\[ \exp(i \mu \pi) - \exp(-i \mu \pi) = 0. \quad (i) \]

(b) Let \( \mu = \nu + i \sigma \) and use Euler’s relation \( \exp(iv \pi) = \cos(v \pi) + i \sin(v \pi) \) to determine the real and imaginary parts of Eq. (i).
(c) By considering the equations found in part (b), show that $v$ is an integer and that $\sigma = 0$. Consequently $\mu$ is real and so is $\lambda$. 
10.2 Fourier Series

Later in this chapter you will find that you can solve many important problems involving partial differential equations, provided that you can express a given function as an infinite sum of sines and/or cosines. In this and the following two sections we explain in detail how this can be done. These trigonometric series are called **Fourier series**; they are somewhat analogous to Taylor series in that both types of series provide a means of expressing quite complicated functions in terms of certain familiar elementary functions.

We begin with a series of the form

\[
\frac{a_0}{2} + \sum_{m=1}^{\infty} \left( a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right).
\]  

On the set of points where the series (1) converges, it defines a function \( f \), whose value at each point is the sum of the series for that value of \( x \). In this case the series (1) is said to be the Fourier series for \( f \). Our immediate goals are to determine what functions can be represented as a sum of a Fourier series and to find some means of computing the coefficients in the series corresponding to a given function. The first term in the series (1) is written as \( \frac{a_0}{2} \) rather than simply as \( a_0 \) to simplify a formula for the coefficients that we derive below. Besides their association with the method of separation of variables and partial differential equations, Fourier series are also useful in various other ways, such as in the analysis of mechanical or electrical systems acted on by periodic external forces.

**Periodicity of the Sine and Cosine Functions.** To discuss Fourier series it is necessary to develop certain properties of the trigonometric functions \( \sin \left( \frac{m\pi x}{L} \right) \) and \( \cos \left( \frac{m\pi x}{L} \right) \), where \( m \) is a positive integer. The first property is their periodic character. A function \( f \) is said to be **periodic** with period \( T > 0 \) if the domain of \( f \) contains \( x + T \) whenever it contains \( x \), and if

\[
f(x + T) = f(x)
\]  

for every value of \( x \). An example of a periodic function is shown in Figure 10.2.1. It follows immediately from the definition that if \( T \) is a period of \( f \), then \( 2T \) is also a period, and so indeed is any integral multiple of \( T \). The smallest value of \( T \) for which Eq. (2) holds is called the **fundamental period** of \( f \). A constant function is a periodic function with an arbitrary period but no fundamental period.
If \( f \) and \( g \) are any two periodic functions with common period \( T \), then their product \( f g \) and any linear combination \( c_1 f + c_2 g \) are also periodic with period \( T \). To prove the latter statement, let 
\[
F(x) = c_1 f(x) + c_2 g(x);
\]
then for any \( x \)
\[
F(x + T) = c_1 f(x + T) + c_2 g(x + T) = c_1 f(x) + c_2 g(x) = F(x). \tag{3}
\]

Moreover, it can be shown that the sum of any finite number, or even the sum of a convergent infinite series, of functions of period \( T \) is also periodic with period \( T \).

In particular, the functions \( \sin(mx/L) \) and \( \cos(mx/L) \) \( m = 1, 2, 3, \ldots \), are periodic with fundamental period \( T = 2L/m \). To see this, recall that \( \sin x \) and \( \cos x \) have fundamental period \( 2\pi \) and that \( \sin \alpha x \) and \( \cos \alpha x \) have fundamental period \( 2\pi/\alpha \). If we choose \( \alpha = m\pi/L \), then the period \( T \) of \( \sin(mx/L) \) and \( \cos(mx/L) \) is given by \( T = 2\pi L / m\pi = 2L/m \).

Note also that, since every positive integral multiple of a period is also a period, each of the functions \( \sin(mx/L) \) and \( \cos(mx/L) \) has the common period \( 2L \).

**Orthogonality of the Sine and Cosine Functions.** To describe a second essential property of the functions \( \sin(mx/L) \) and \( \cos(mx/L) \), we generalize the concept of orthogonality of vectors (see Section 7.2). The standard inner product \( (u, v) \) of two real-valued functions \( u \) and \( v \) on the interval \( \alpha \leq x \leq \beta \) is defined by
\[
(u, v) = \int_\alpha^\beta u(x)v(x)dx. \tag{4}
\]

The functions \( u \) and \( v \) are said to be **orthogonal** on \( \alpha \leq x \leq \beta \) if their inner product is zero—that is, if
\[
\int_\alpha^\beta u(x)v(x)dx = 0. \tag{5}
\]

A set of functions is said to be **mutually orthogonal** if each distinct pair of functions in the set is orthogonal.

The functions \( \sin(mx/L) \) and \( \cos(mx/L) \) \( m = 1, 2, \ldots \) form a mutually orthogonal set of functions on the interval \( -L \leq x \leq L \). In fact, they satisfy the following orthogonality relations:
\[
\int_{-L}^{L} \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx = \begin{cases} 
0, & m \neq n, \\
L, & m = n; 
\end{cases} \tag{6}
\]
\[
\int_{-L}^{L} \cos \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx = 0, \quad \text{all } m, n; \tag{7}
\]
\[
\int_{-L}^{L} \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx = \begin{cases} 
0, & m \neq n, \\
L, & m = n. 
\end{cases} \tag{8}
\]
These results can be obtained by direct integration. For example, to derive Eq. (8), note that

\[
\int_{-L}^{L} \sin \frac{m \pi x}{L} \sin \frac{n \pi x}{L} \, dx = \frac{1}{2} \int_{-L}^{L} \left[ \cos \frac{(m-n) \pi x}{L} - \cos \frac{(m+n) \pi x}{L} \right] \, dx \\
= \frac{1}{2} \left[ \frac{\sin \left(\frac{(m-n) \pi x}{L}\right)}{m-n} - \frac{\sin \left(\frac{(m+n) \pi x}{L}\right)}{m+n} \right]_{-L}^{L} \\
= 0
\]

as long as \( m + n \) and \( m - n \) are not zero. Since \( m \) and \( n \) are positive, \( m + n \neq 0 \). On the other hand, if \( m - n = 0 \), then \( m = n \), and the integral must be evaluated in a different way. In this case

\[
\int_{-L}^{L} \sin \frac{m \pi x}{L} \sin \frac{n \pi x}{L} \, dx = \int_{-L}^{L} \left( \sin \frac{m \pi x}{L} \right)^2 \, dx \\
= \frac{1}{2} \int_{-L}^{L} \left[ 1 - \cos \frac{2m \pi x}{L} \right] \, dx \\
= \frac{1}{2} \left[ x - \frac{\sin \left(\frac{2m \pi x}{L}\right)}{2m \pi} \right]_{-L}^{L} \\
= L.
\]

This establishes Eq. (8); Eqs. (6) and (7) can be verified by similar computations.

**The Euler–Fourier Formulas.** Now let us suppose that a series of the form (1) converges, and let us call its sum \( f(x) \):

\[
f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left( a_m \cos \frac{m \pi x}{L} + b_m \sin \frac{m \pi x}{L} \right). \quad (9)
\]

The coefficients \( a_m \) and \( b_m \) can be related to \( f(x) \) as a consequence of the orthogonality conditions (6), (7), and (8). First multiply Eq. (9) by \( \cos \left(\frac{n \pi x}{L}\right) \), where \( n \) is a fixed positive integer (\( n > 0 \)), and integrate with respect to \( x \) from \(-L\) to \( L\). Assuming that the integration can be legitimately carried out term by term, we obtain

\[
\int_{-L}^{L} f(x) \cos \frac{n \pi x}{L} \, dx = \frac{a_0}{2} \int_{-L}^{L} \cos \frac{n \pi x}{L} \, dx + \sum_{m=1}^{\infty} a_m \int_{-L}^{L} \cos \frac{m \pi x}{L} \cos \frac{n \pi x}{L} \, dx \\
+ \sum_{m=1}^{\infty} b_m \int_{-L}^{L} \sin \frac{m \pi x}{L} \cos \frac{n \pi x}{L} \, dx. \quad (10)
\]

Keeping in mind that \( n \) is fixed whereas \( m \) ranges over the positive integers, it follows from the orthogonality relations (6) and (7) that the only nonzero term on the right side of Eq. (10) is the one for which \( m = n \) in the first summation. Hence

\[
\int_{-L}^{L} f(x) \cos \frac{n \pi x}{L} \, dx = L a_n, \quad n = 1, 2, \ldots \quad (11)
\]

To determine \( a_0 \) we can integrate Eq. (9) from \(-L\) to \( L\), obtaining
since each integral involving a trigonometric function is zero. Thus

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{m\pi x}{L} \, dx, \quad n = 0, 1, 2, \ldots$$  \hspace{1cm} (13)$$

By writing the constant term in Eq. (9) as \(a_0/2\), it is possible to compute all the \(a_n\) from Eq. (13). Otherwise, a separate formula would have to be used for \(a_0\).

A similar expression for \(b_n\) may be obtained by multiplying Eq. (9) by \(\sin (m\pi x/L)\), integrating termwise from \(-L\) to \(L\), and using the orthogonality relations (7) and (8); thus

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{m\pi x}{L} \, dx, \quad n = 1, 2, 3, \ldots$$  \hspace{1cm} (14)$$

Equations (13) and (14) are known as the Euler–Fourier formulas for the coefficients in a Fourier series. Hence, if the series (9) converges to \(f(x)\), and if the series can be integrated term by term, then the coefficients must be given by Eqs. (13) and (14).

Note that Eqs. (13) and (14) are explicit formulas for \(a_n\) and \(b_n\) in terms of \(f\), and that the determination of any particular coefficient is independent of all the other coefficients. Of course, the difficulty in evaluating the integrals in Eqs. (13) and (14) depends very much on the particular function \(f\) involved.

Note also that the formulas (13) and (14) depend only on the values of \(f(x)\) in the interval \(-L \leq x \leq L\). Since each of the terms in the Fourier series (9) is periodic with period \(2L\), the series converges for all \(x\) whenever it converges in \(-L \leq x \leq L\), and its sum is also a periodic function with period \(2L\). Hence \(f(x)\) is determined for all \(x\) by its values in the interval \(-L \leq x \leq L\).

It is possible to show (see Problem 27) that if \(g\) is periodic with period \(T\), then every integral of \(g\) over an interval of length \(T\) has the same value. If we apply this result to the Euler–Fourier formulas (13) and (14), it follows that the interval of integration, \(-L \leq x \leq L\), can be replaced, if it is more convenient to do so, by any other interval of length \(2L\).

**Example:** Assume that there is a Fourier series converging to the function \(f\) defined by

$$f(x) = \begin{cases} 
-x, & -2 \leq x < 0, \\
x, & 0 \leq x < 2; \\
0, & \text{otherwise}.
\end{cases}$$  \hspace{1cm} (15)$$

Determine the coefficients in this Fourier series.

This function represents a triangular wave (see Figure 10.2.2) and is periodic with period 4. Thus in this case \(L = 2\), and the Fourier series has the form

$$\int_{-L}^{L} f(x) \, dx = \frac{a_0}{2} \int_{-L}^{L} dx + \sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos \frac{m\pi x}{L} \, dx + \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin \frac{m\pi x}{L} \, dx$$

$$= La_0,$$  \hspace{1cm} (12)$$
where the coefficients are computed from Eqs. (13) and (14) with \( L = 2 \). Substituting for \( f(x) \) in Eq. (13) with \( m = 0 \), we have

\[
a_0 = \frac{1}{2} \int_{-2}^{0} (-x) \, dx + \frac{1}{2} \int_{0}^{2} x \, dx = 1 + 1 = 2. \tag{17}
\]

For \( m > 0 \), Eq. (13) yields

\[
a_m = \frac{1}{2} \left[ -\frac{2}{m\pi} x \sin \frac{m\pi x}{2} - \left( \frac{2}{m\pi} \right)^2 \cos \frac{m\pi x}{2} \right]_{-2}^{0} + \frac{1}{2} \left[ \frac{2}{m\pi} x \sin \frac{m\pi x}{2} + \left( \frac{2}{m\pi} \right)^2 \cos \frac{m\pi x}{2} \right]_{0}^{2}
\]

These integrals can be evaluated through integration by parts, with the result that

\[
a_m = \frac{1}{2} \left[ -\frac{2}{m\pi} x \sin \frac{m\pi x}{2} - \left( \frac{2}{m\pi} \right)^2 \cos \frac{m\pi x}{2} \right]_{-2}^{0} + \frac{1}{2} \left[ \frac{2}{m\pi} x \sin \frac{m\pi x}{2} + \left( \frac{2}{m\pi} \right)^2 \cos \frac{m\pi x}{2} \right]_{0}^{2}
\]

\[
= \frac{1}{2} \left[ -\left( \frac{2}{m\pi} \right)^2 + \left( \frac{2}{m\pi} \right)^2 \cos m\pi + \left( \frac{2}{m\pi} \right)^2 \cos m\pi - \left( \frac{2}{m\pi} \right)^2 \right]
\]

\[
= \frac{4}{(m\pi)^2} (\cos m\pi - 1), \quad m = 1, 2, \ldots
\]

Finally, from Eq. (14) it follows in a similar way that

\[
b_m = 0, \quad m = 1, 2, \ldots \tag{19}
\]

By substituting the coefficients from Eqs. (17), (18), and (19) in the series (16), we obtain the Fourier series for \( f \):

**Figure 10.2.2** Triangular wave.
EXAMPLE 2

Let

\[ f(x) = \begin{cases} 
0, & -3 < x < -1, \\
1, & -1 < x < 1, \\
0, & 1 < x < 3 
\end{cases} \]  

and suppose that \( f(x + 6) = f(x) \); see Figure 10.2.3. Find the coefficients in the Fourier series for \( f \).

Since \( f \) has period 6, it follows that \( L = 3 \) in this problem. Consequently, the Fourier series for \( f \) has the form

\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{3} + b_n \sin \frac{n\pi x}{3} \right), \]  

where the coefficients \( a_n \) and \( b_n \) are given by Eqs. (13) and (14) with \( L = 3 \). We have

\[ a_0 = \frac{1}{3} \int_{-3}^{3} f(x)dx = \frac{1}{3} \int_{-1}^{1} dx = \frac{2}{3}. \]

Similarly,

\[ a_n = \frac{1}{3} \int_{-1}^{1} \cos \frac{n\pi x}{3} dx = \frac{1}{n\pi} \sin \frac{n\pi x}{3} \bigg|_{-1}^{1} = \frac{2}{n\pi} \sin \frac{n\pi}{3}, \quad n = 1, 2, ..., \]  

and

\[ b_n = \frac{1}{3} \int_{-1}^{1} \sin \frac{n\pi x}{3} dx = -\frac{1}{n\pi} \cos \frac{n\pi x}{3} \bigg|_{-1}^{1} = 0, \quad n = 1, 2, ... \]

Thus the Fourier series for \( f \) is

\[ f(x) = 1 - \frac{8}{\pi^2} \left( \cos \frac{\pi x}{2} + \frac{1}{3^2} \cos \frac{3\pi x}{2} + \frac{1}{5^2} \cos \frac{5\pi x}{2} + ... \right) \]

\[ = 1 - \frac{8}{\pi^2} \sum_{m=1,3,5,...}^{\infty} \frac{\cos(m\pi x/2)}{m^2} \]

\[ = 1 - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(2n - 1)\pi x/2}{(2n - 1)^2}. \]
Consider again the function in Example 1 and its Fourier series (20). Investigate the speed with which the series converges. In particular, determine how many terms are needed so that the error is no greater than 0.01 for all \( x \).

The \( m \)th partial sum in this series,

\[
s_m(x) = 1 - \frac{8}{\pi^2} \sum_{n=1}^{m} \frac{\cos((2n-1)\pi x/2)}{(2n-1)^2},
\]

(27) can be used to approximate the function \( f \). The coefficients diminish as \( (2n-1)^{-2} \), so the series converges fairly rapidly. This is borne out by Figure 10.2.4, where the partial sums for \( m = 1 \) and \( m = 2 \) are plotted. To investigate the convergence in more detail we can consider the error \( e_m(x) = f(x) - s_m(x) \). Figure 10.2.5 shows a plot of \( |e_6(x)| \) versus \( x \) for \( 0 \leq x \leq 2 \). Observe that \( |e_6(x)| \) is greatest at the points \( x = 0 \) and \( x = 2 \) where the graph of \( f(x) \) has corners. It is more difficult for the series to approximate the function near these points, resulting in a larger error there for a given \( m \). Similar graphs are obtained for other values of \( m \).

**Figure 10.2.4** Partial sums in the Fourier series, Eq. (20), for the triangular wave.
Once you realize that the maximum error always occurs at $x = 0$ or $x = 2$, you can obtain a uniform error bound for each $m$ simply by evaluating $|e_m(x)|$ at one of these points. For example, $m = 6$ we have $e_6(2) = 0.03370$, so $|e_6(x)| < 0.034$ for $0 \leq x \leq 2$ and consequently for all $x$. Table 2.1 shows corresponding data for other values of $m$; these data are plotted in Figure 10.2.6. From this information you can begin to estimate the number of terms that are needed in the series in order to achieve a given level of accuracy in the approximation. For example, to guarantee that $|e_m(x)| \leq 0.01$ we need to choose $m = 21$.

**Table 10.2.1** Values of the error $e_m(2)$ for the triangular wave

<table>
<thead>
<tr>
<th>$m$</th>
<th>$e_m(2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.09937</td>
</tr>
<tr>
<td>4</td>
<td>0.05040</td>
</tr>
<tr>
<td>6</td>
<td>0.03370</td>
</tr>
<tr>
<td>10</td>
<td>0.02025</td>
</tr>
<tr>
<td>15</td>
<td>0.01350</td>
</tr>
<tr>
<td>20</td>
<td>0.01013</td>
</tr>
<tr>
<td>25</td>
<td>0.00810</td>
</tr>
</tbody>
</table>
In this book Fourier series appear mainly as a means of solving certain problems in partial differential equations. However, such series have much wider application in science and engineering and, in general, are valuable tools in the investigation of periodic phenomena. A basic problem is to resolve an incoming signal into its harmonic components, which amounts to constructing its Fourier series representation. In some frequency ranges the separate terms correspond to different colors or to different audible tones. The magnitude of the coefficient determines the amplitude of each component. This process is referred to as spectral analysis.

**PROBLEMS**

In each of Problems 1 through 8 determine whether the given function is periodic. If so, find its fundamental period.

1. \( \sin 5x \)

2. \( \cos 2\pi x \)

3. \( \sinh 2x \)

4. \( \sin \frac{\pi x}{L} \)
5. \( \tan \pi x \)

6. \( x^2 \)

7. 
\[
f(x) = \begin{cases} 
0, & 2n - 1 \leq x < 2n, \\
1, & 2n \leq x < 2n + 1; 
\end{cases} \quad n = 0, \pm 1, \pm 2, \ldots
\]

8. 
\[
f(x) = \begin{cases} 
(-1)^n, & 2n - 1 \leq x < 2n, \\
1, & 2n \leq x < 2n + 1; 
\end{cases} \quad n = 0, \pm 1, \pm 2, \ldots
\]

9. If \( f(x) = -x \) for \(-L < x < L\), and if \( f(x + 2L) = f(x) \), find a formula for \( f(x) \) in the interval \( L < x < 2L \); in the interval \(-3L < x < 2L\).

10. If \( f(x) = \begin{cases} 
-x + 1, & -1 < x < 0, \\
x, & 0 < x < 1, \\
1 < x < 2;
\end{cases} \) and if \( f(x + 2) = f(x) \), find a formula for \( f(x) \) in the interval \( 1 < x < 2 \); in the interval \( 8 < x < 9 \).

11. If \( f(x) = L - x \) for \( 0 < x < 2L \), and if \( f(x + 2L) = f(x) \), find a formula for \( f(x) \) in the interval \(-L < x < 0\).

12. Verify Eqs. (6) and (7) in this section by direct integration.

In each of Problems 13 through 18:

(a) Sketch the graph of the given function for three periods.

(b) Find the Fourier series for the given function.

13. \( f(x) = -x, \quad -L \leq x < L; \quad f(x + 2L) = f(x) \)
14. 
\[ f(x) = \begin{cases} 1, & -L \leq x < 0, \\ 0, & 0 \leq x < L; \end{cases} \quad f(x + 2L) = f(x) \]

15. 
\[ f(x) = \begin{cases} x, & -\pi \leq x < 0, \\ 0, & 0 \leq x < \pi; \end{cases} \quad f(x + 2\pi) = f(x) \]

16. 
\[ f(x) = \begin{cases} x + 1, & -1 \leq x < 0, \\ 1 - x, & 0 \leq x < 1; \end{cases} \quad f(x + 2) = f(x) \]

17. 
\[ f(x) = \begin{cases} x + L, & -L \leq x \leq 0, \\ L, & 0 < x < L; \end{cases} \quad f(x + 2L) = f(x) \]

18. 
\[ f(x) = \begin{cases} 0, & -2 \leq x \leq -1, \\ x, & -1 < x < 1, \\ 0, & 1 \leq x < 2; \end{cases} \quad f(x + 4) = f(x) \]

Just Ask!

In each of Problems 19 through 24:

(a) Sketch the graph of the given function for three periods.

(b) Find the Fourier series for the given function.

(c) Plot \( s_m(x) \) versus \( x \) for \( m = 5, 10, \) and 20.

(d) Describe how the Fourier series seems to be converging.

19. 
\[ f(x) = \begin{cases} -1, & -2 \leq x < 0, \\ 1, & 0 \leq x < 2; \end{cases} \quad f(x + 4) = f(x) \]
20. \( f(x) = x, \quad -1 \leq x < 1; \quad f(x + 2) = f(x) \)

21. \( f(x) = x^2/2, \quad -2 \leq x \leq 2; \quad f(x + 4) = f(x) \)

22. \( f(x) = \begin{cases} x + 2, & -2 \leq x < 0, \\ 2 - 2x, & 0 \leq x < 2; \end{cases} \quad f(x + 4) = f(x) \)

23. \( f(x) = \begin{cases} -\frac{1}{2}x, & -2 \leq x < 0, \\ 2x - \frac{1}{2}x^2, & 0 \leq x < 2; \end{cases} \quad f(x + 4) = f(x) \)

24. \( f(x) = \begin{cases} 0, & -3 \leq x < 0, \\ x^2(3-x), & 0 \leq x < 3; \end{cases} \quad f(x + 6) = f(x) \)

25. Consider the function \( f \) defined in Problem 21 and let \( e_m(x) = f(x) - s_m(x) \). Plot \( |e_m(x)| \) versus \( x \) for \( 0 \leq x \leq 2 \) for several values of \( m \). Find the smallest value of \( m \) for which \( |e_m(x)| \leq 0.01 \) for all \( x \).

26. Consider the function \( f \) defined in Problem 24 and let \( e_m(x) = f(x) - s_m(x) \). Plot \( |e_m(x)| \) versus \( x \) for \( 0 \leq x \leq 3 \) for several values of \( m \). Find the smallest value of \( m \) for which \( |e_m(x)| \leq 0.1 \) for all \( x \).

27. Suppose that \( g \) is an integrable periodic function with period \( T \).

(a) If \( 0 \leq a \leq T \), show that

\[
\int_0^T g(x)\,dx = \int_a^{a+T} g(x)\,dx.
\]

*Hint:* Show first that \( \int_0^a g(x)\,dx = \int_T^{a+T} g(x)\,dx \). Consider the change of variable \( s = x - T \) in the second integral.
(b) Show that for any value of $a$, not necessarily in $0 \leq a \leq T$,
\[ \int_0^T g(x) \, dx = \int_a^{a+T} g(x) \, dx. \]

(c) Show that for any values of $a$ and $b$,
\[ \int_a^{a+T} g(x) \, dx = \int_b^{b+T} g(x) \, dx. \]

28. If $f$ is differentiable and is periodic with period $T$, show that $f'$ is also periodic with period $T$. Determine whether
\[ F(x) = \int_0^x f(t) \, dt \]
is always periodic.

29. In this problem we indicate certain similarities between three-dimensional geometric vectors and Fourier series.

(a) Let $v_1, v_2, v_3$ be a set of mutually orthogonal vectors in three dimensions, and let $u$ be any three-dimensional vector. Show that
\[ u = a_1 v_1 + a_2 v_2 + a_3 v_3, \]  
where
\[ a_i = \frac{u \cdot v_i}{v_i \cdot v_i}, \quad i = 1, 2, 3. \]

Show that $a_i$ can be interpreted as the projection of $u$ in the direction of $v_i$ divided by the length of $v_i$.

(b) Define the inner product $(u, v)$ by
\[ (u, v) = \int_L^L u(x) v(x) \, dx. \]  
Also let
Show that Eq. (10) can be written in the form

\[
(f, \phi_n) = \frac{a_0}{2} (\phi_0, \phi_n) + \sum_{m=1}^{\infty} a_m (\phi_m, \phi_n) + \sum_{m=1}^{\infty} b_m (\psi_m, \phi_n). \tag{v}
\]

Use Eq. (v) and the corresponding equation for \((f, \psi_n)\), together with the orthogonality relations, to show that

\[
a_n = \frac{(f, \phi_n)}{(\phi_n, \phi_n)}, \quad n = 0, 1, 2, \ldots; \quad b_n = \frac{(f, \psi_n)}{(\psi_n, \psi_n)}, \quad n = 1, 2, \ldots. \tag{vi}
\]

Note the resemblance between Eqs. (vi) and Eq. (ii). The functions \(\phi_n\) and \(\psi_n\) play a role for functions similar to that of the orthogonal vectors \(\mathbf{v}_1\), \(\mathbf{v}_2\), and \(\mathbf{v}_3\) in three-dimensional space. The coefficients \(a_n\) and \(b_n\) can be interpreted as projections of the function \(f\) onto the base functions \(\phi_n\) and \(\psi_n\).

Observe also that any vector in three dimensions can be expressed as a linear combination of three mutually orthogonal vectors. In a somewhat similar way, any sufficiently smooth function defined on \(-L \leq x \leq L\) can be expressed as a linear combination of the mutually orthogonal functions \(\cos(n \pi x / L)\) and \(\sin(n \pi x / L)\), that is, as a Fourier series.
10.3 The Fourier Convergence Theorem

In the preceding section we showed that if the Fourier series

\[
\frac{a_0}{2} + \sum_{m=1}^{\infty} \left( a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right)
\]  

(1)

converges and thereby defines a function \( f \), then \( f \) is periodic with period \( 2L \), and the coefficients \( a_m \) and \( b_m \) are related to \( f(x) \) by the Euler–Fourier formulas:

\[
a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{m\pi x}{L} \, dx, \quad m = 0, 1, 2, \ldots
\]

(2)

\[
b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{m\pi x}{L} \, dx, \quad m = 1, 2, \ldots
\]

(3)

In this section we suppose that a function \( f \) is given. If this function is periodic with period \( 2L \) and integrable on the interval \([ -L, L ]\), then a set of coefficients \( a_m \) and \( b_m \) can be computed from Eqs. (2) and (3), and a series of the form (1) can be formally constructed. The question is whether this series converges for each value of \( x \) and, if so, whether its sum is \( f(x) \). Examples have been discovered showing that the Fourier series corresponding to a function \( f \) may not converge to \( f(x) \) or may even diverge. Functions whose Fourier series do not converge to the value of the function at isolated points are easily constructed, and examples will be presented later in this section. Functions whose Fourier series diverge at one or more points are more pathological, and we will not consider them in this book.

To guarantee convergence of a Fourier series to the function from which its coefficients were computed, it is essential to place additional conditions on the function. From a practical point of view, such conditions should be broad enough to cover all situations of interest, yet simple enough to be easily checked for particular functions. Through the years several sets of conditions have been devised to serve this purpose.

Before stating a convergence theorem for Fourier series, we define a term that appears in the theorem. A function \( f \) is said to be **piecewise continuous** on an interval \( a \leq x \leq b \) if the interval can be partitioned by a finite number of points \( a = x_0 < x_1 < \cdots < x_n = b \) so that

1. \( f \) is continuous on each open subinterval \( x_{i-1} < x < x_i \).

2. \( f \) approaches a finite limit as the endpoints of each subinterval are approached from within the subinterval.
The graph of a piecewise continuous function is shown in Figure 10.3.1.

![Figure 10.3.1 A piecewise continuous function.](image)

The notation \( f(c+) \) is used to denote the limit of \( f(x) \) as \( x \to c \) from the right; similarly, \( f(c-) \) denotes the limit of \( f(x) \) as \( x \) approaches \( c \) from the left.

Note that it is not essential that the function even be defined at the partition points \( x_i \). For example, in the following theorem we assume that \( f' \) is piecewise continuous; but certainly \( f' \) does not exist at those points where \( f \) itself is discontinuous. It is also not essential that the interval be closed; it may also be open, or open at one end and closed at the other.

**Theorem 10.3.1** Suppose that \( f \) and \( f' \) are piecewise continuous on the interval \( -L \leq x < L \). Further, suppose that \( f \) is defined outside the interval \( -L \leq x < L \) so that it is periodic with period \( 2L \). Then \( f \) has a Fourier series

\[
f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left( a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right),
\]

whose coefficients are given by Eqs. (2) and (3). The Fourier series converges to \( f(x) \) at all points where \( f \) is continuous, and to \( [f(x+) + f(x-)]/2 \) at all points where \( f \) is discontinuous.

Note that \( [f(x+) + f(x-)]/2 \) is the mean value of the right- and left-hand limits at the point \( x \). At any point where \( f \) is continuous, \( f(x+) = f(x-) = f(x) \). Thus it is correct to say that the Fourier series converges to \( [f(x+) + f(x-)]/2 \) at all points. Whenever we say that a Fourier series converges to a function \( f \), we always mean that it converges in this sense.

It should be emphasized that the conditions given in this theorem are only sufficient for the convergence of a Fourier series; they are by no means necessary. Nor are they the most general sufficient conditions that have been discovered. In spite of this, the proof of the theorem is fairly difficult and we do not discuss it here. * Under more restrictive conditions a much simpler convergence proof is possible; see Problem 18.

To obtain a better understanding of the content of the theorem, it is helpful to consider some classes of functions that fail to satisfy the assumed conditions. Functions that are not included in...
the theorem are primarily those with infinite discontinuities in the interval \([-L, L]\), such as \(1/x^2\) as \(x \to 0\), or \(\ln |x-L|\) as \(x \to L\). Functions having an infinite number of jump discontinuities in this interval are also excluded; however, such functions are rarely encountered.

It is noteworthy that a Fourier series may converge to a sum that is not differentiable, or even continuous, in spite of the fact that each term in the series (4) is continuous, and even differentiable infinitely many times. The example below is an illustration of this, as is Example 2 in Section 10.2.

**Example 1**

Let

\[
    f(x) = \begin{cases} 
        0, & -L < x < 0, \\
        L, & 0 < x < L. 
    \end{cases}
\]  

(5)

and let \(f\) be defined outside this interval so that \(f(x + 2L) = f(x)\) for all \(x\). We will temporarily leave open the definition of \(f\) at the points \(x = 0, \pm L\), except to say that its value must be finite. Find the Fourier series for this function and determine where it converges.

The equation \(y = f(x)\) has the graph shown in Figure 10.3.2, extended to infinity in both directions. It can be thought of as representing a square wave. The interval \([-L, L]\) can be partitioned to give the two open subintervals \((-L, 0)\) and \((0, L)\). In \((0, L)\), \(f(x) = L\) and \(f'(x) = 0\). Clearly, both \(f\) and \(f'\) are continuous and furthermore have limits as \(x \to 0\) from the right and as \(x \to L\) from the left. The situation in \((-L, 0)\) is similar. Consequently, both \(f\) and \(f'\) are piecewise continuous on \([-L, L]\), so \(f\) satisfies the conditions of Theorem 10.3.1. If the coefficients \(a_m\) and \(b_m\) are computed from Eqs. (2) and (3), the convergence of the resulting Fourier series to \(f(x)\) is ensured at all points where \(f\) is continuous. Note that the values of \(a_m\) and \(b_m\) are the same regardless of the definition of \(f\) at its points of discontinuity. This is true because the value of an integral is unaffected by changing the value of the integrand at a finite number of points. From Eq. (2),

![Figure 10.3.2 Square wave.](image-url)
Similarly, from Eq. (3),

\[ a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{m\pi x}{L} \, dx = \int_{0}^{L} \cos \frac{m\pi x}{L} \, dx \]

\[ a_m = 0, \ m \neq 0. \]

Hence

\[ b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{m\pi x}{L} \, dx = \int_{0}^{L} \sin \frac{m\pi x}{L} \, dx \]

\[ b_m = \begin{cases} 0, & m \text{ even;} \\ \frac{2L}{m\pi}, & m \text{ odd.} \end{cases} \]

At the points \( x = 0, \pm nL \), where the function \( f \) in the example is not continuous, all terms in the series after the first vanish and the sum is \( L/2 \). This is the mean value of the limits from the right and left, as it should be. Thus we might as well define \( f \) at these points to have the value \( L/2 \). If we choose to define it otherwise, the series still gives the value \( L/2 \) at these points, since none of the preceding calculations is altered in any detail; it simply does not converge to the function at those points unless \( f \) is defined to have this value. This illustrates the possibility that the Fourier series corresponding to a function may not converge to it at points of discontinuity unless the function is suitably defined at such points.

The manner in which the partial sums

\[ s_n(x) = \frac{L}{2} + \frac{2L}{\pi} \left( \sin \frac{\pi x}{L} + \frac{1}{3} \sin \frac{3\pi x}{L} + \frac{1}{5} \sin \frac{5\pi x}{L} + \ldots \right) \]

\[ s_n(x) = \frac{L}{2} + \frac{2L}{\pi} \sum_{m=1,3,5,\ldots}^{\infty} \frac{\sin(m\pi x/L)}{m} \]

\[ s_n(x) = \frac{L}{2} + \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x/L)}{2n-1} \]

of the Fourier series (6) converge to \( f(x) \) is indicated in Figure 10.3.3, where \( L \) has been chosen to be 1 and the graph of \( s_8(x) \) is plotted. The figure suggests that at points where \( f \) is continuous the partial sums do approach \( f(x) \) as \( n \) increases. However, in the neighborhood of points of discontinuity, such as \( x = 0 \) and \( x = L \), the partial sums do not converge smoothly to the mean value. Instead they tend to overshoot the mark at each end of the jump, as though they cannot quite accommodate themselves to the sharp turn required at this point. This behavior is
typical of Fourier series at points of discontinuity and is known as the Gibbs phenomenon.

Figure 10.3.3  The partial sum \( s_n(x) \) in the Fourier series, Eq. (6), for the square wave.

Additional insight is attained by considering the error \( e_n(x) = f(x) - s_n(x) \). Figure 10.3.4 shows a plot of \( |e_n(x)| \) versus \( x \) for \( n = 8 \) and for \( L = 1 \). The least upper bound of \( |e_8(x)| \) is 0.5 and is approached as \( x \to 0 \) and as \( x \to 1 \). As \( n \) increases, the error decreases in the interior of the interval [where \( f(x) \) is continuous], but the least upper bound does not diminish with increasing \( n \). Thus one cannot uniformly reduce the error throughout the interval by increasing the number of terms.

Figure 10.3.4  A plot of the error \( |e_8(x)| \) versus \( x \) for the square wave.

Figures 10.3.3 and 10.3.4 also show that the series in this example converges more slowly than the one in Example 1 in Section 10.2. This is due to the fact that the coefficients in the series (6) are proportional only to \( 1/(2n - 1) \).

PROBLEMS
In each of Problems 1 through 6 assume that the given function is periodically extended outside the original interval.

(a) Find the Fourier series for the extended function.

(b) Sketch the graph of the function to which the series converges for three periods.

1. \[ f(x) = \begin{cases} -1, & -1 \leq x < 0, \\ 1, & 0 \leq x < 1 \end{cases} \]

2. \[ f(x) = \begin{cases} 0, & -\pi \leq x < 0, \\ x, & 0 \leq x < \pi \end{cases} \]

3. \[ f(x) = \begin{cases} L + x, & -L \leq x < 0, \\ L - x, & 0 \leq x < L \end{cases} \]

4. \[ f(x) = 1 - x^2, \quad -1 \leq x < 1 \]

5. \[ f(x) = \begin{cases} 0, & -\pi \leq x < -\pi/2, \\ 1, & -\pi/2 \leq x < \pi/2, \\ 0, & \pi/2 \leq x < \pi \end{cases} \]

6. \[ f(x) = \begin{cases} 0, & -1 \leq x < 0, \\ x^2, & 0 \leq x < 1 \end{cases} \]

In each of Problems 7 through 12 assume that the given function is periodically extended outside the original interval.

(a) Find the Fourier series for the given function.
(b) Let \( e_n(x) = f(x) - s_n(x) \). Find the least upper bound or the maximum value (if it exists) of \(|e_n(x)|\) for \( n = 10, 20, \) and 40.

(c) If possible, find the smallest \( n \) for which \(|e_n(x)| \leq 0.01\) for all \( x \).

7. \( f(x) = \begin{cases} x, & -\pi \leq x < 0, \\ 0, & 0 \leq x < \pi; \end{cases} \quad f(x + 2\pi) = f(x) \) (see Section 10.2, Problem 15)

8. \( f(x) = \begin{cases} x + 1, & -1 \leq x < 0, \\ 1 - x, & 0 \leq x < 1; \end{cases} \quad f(x + 2) = f(x) \) (see Section 10.2, Problem 16)

9. \( f(x) = x, \quad -1 \leq x < 1; \quad f(x + 2) = f(x) \) (see Section 10.2, Problem 20)

10. \( f(x) = \begin{cases} x + 2, & -2 \leq x < 0, \\ 2 - 2x, & 0 \leq x < 2; \end{cases} \quad f(x + 4) = f(x) \) (see Section 10.2, Problem 22)

11. \( f(x) = \begin{cases} 0, & -1 \leq x < 0, \\ x^2, & 0 \leq x < 1; \end{cases} \quad f(x + 2) = f(x) \) (see Problem 6)

12. \( f(x) = x - x^3, \quad -1 \leq x < 1; \quad f(x + 2) = f(x) \)

Just Ask!

**Periodic Forcing Terms.** In this chapter we are concerned mainly with the use of Fourier series to solve boundary value problems for certain partial differential equations. However, Fourier series are also useful in many other situations where periodic phenomena occur. Problems 13 through 16 indicate how they can be employed to solve initial value problems with periodic forcing terms.

13. Find the solution of the initial value problem

\[ y'' + \omega^2 y = \sin nt, \quad y(0) = 0, \quad y'(0) = 0, \]
where \( n \) is a positive integer and \( \omega^2 \neq n^2 \). What happens if \( \omega^2 = n^2 \)?

14. Find the formal solution of the initial value problem

\[
y'' + \omega^2 y = \sum_{n=1}^{\infty} b_n \sin nt, \quad y(0) = 0, \quad y'(0) = 0,
\]

where \( \omega > 0 \) is not equal to a positive integer. How is the solution altered if \( \omega = m \), where \( m \) is a positive integer?

15. Find the formal solution of the initial value problem

\[
y'' + \omega^2 y = f(t), \quad y(0) = 0, \quad y'(0) = 0,
\]

where \( f \) is periodic with period \( 2\pi \) and

\[
f(t) = \begin{cases} 
1, & 0 < t < \pi; \\
0, & t = 0, \pi, 2\pi; \\
-1, & \pi < t < 2\pi.
\end{cases}
\]

See Problem 1.

16. Find the formal solution of the initial value problem

\[
y'' + \omega^2 y = f(t), \quad y(0) = 1, \quad y'(0) = 0,
\]

where \( f \) is periodic with period 2 and

\[
f(t) = \begin{cases} 
1 - t, & 0 \leq t < 1; \\
-1 + t, & 1 \leq t < 2.
\end{cases}
\]

See Problem 8.

17. Assuming that

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right), \quad (i)
\]

show formally that
This relation between a function \( f \) and its Fourier coefficients is known as Parseval's equation. This relation is very important in the theory of Fourier series; see Problem 9 in Section 11.6.

**Hint:** Multiply Eq. (i) by \( f(x) \), integrate from \(-L\) to \(L\), and use the Euler-Fourier formulas.

18. This problem indicates a proof of convergence of a Fourier series under conditions more restrictive than those in Theorem 10.3.1.

(a) If \( f \) and \( f' \) are piecewise continuous on \(-L \leq x < L\), and if \( f \) is periodic with period \(2L\), show that \( na_n \) and \( nb_n \) are bounded as \( n \to \infty \).

**Hint:** Use integration by parts.

(b) If \( f \) is continuous on \(-L \leq x < L\) and periodic with period \(2L\), and if \( f' \) and \( f'' \) are piecewise continuous on \(-L \leq x < L\), show that \( n^2a_n \) and \( n^2b_n \) are bounded as \( n \to \infty \). If \( f \) is continuous on the closed interval, then it is continuous for all \( x \). Why is this important?

**Hint:** Again, use integration by parts.

(c) Using the result of part (b), show that \( \sum_{n=1}^{\infty} |a_n| \) and \( \sum_{n=1}^{\infty} |b_n| \) converge.

(d) From the result in part (c), show that the Fourier series (4) converges absolutely* for all \( x \).

**Acceleration of Convergence.** In the next problem we show how it is sometimes possible to improve the speed of convergence of a Fourier series.

19. Suppose that we wish to calculate values of the function \( g \), where

\[
 g(x) = \sum_{n=1}^{\infty} \frac{(2n-1)}{1+(2n-1)^2} \sin(2n-1)\pi x. \tag{i}
\]

It is possible to show that this series converges, albeit rather slowly. However, observe that for large \( n \) the terms in the series (i) are approximately equal to \( \frac{\sin(2n-1)\pi x}{(2n-1)} \) and that the latter terms are similar to those in the example in the text, Eq. (6).

(a) Show that...
\[ \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x)}{(2n-1)} = \frac{\pi}{2} \left[ f(x) - \frac{1}{2} \right], \quad (\text{ii}) \]

where \( f \) is the square wave in the example with \( L = 1 \).

(b) Subtract Eq. (ii) from Eq. (i) and show that

\[ g(x) = \frac{\pi}{2} \left[ f(x) - \frac{1}{2} \right] - \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x)}{(2n-1)\left[ 1 + (2n-1)^2 \right]}, \quad (\text{iii}) \]

The series (iii) converges much faster than the series (i) and thus provides a better way to calculate values of \( g(x) \).
### 10.4 Even and Odd Functions

Before looking at further examples of Fourier series it is useful to distinguish two classes of functions for which the Euler–Fourier formulas can be simplified. These are even and odd functions, which are characterized geometrically by the property of symmetry with respect to the $y$-axis and the origin, respectively (see Figure 10.4.1).

![Figure 10.4.1](image)

**Figure 10.4.1** (a) An even function. (b) An odd function.

Analytically, $f$ is an **even function** if its domain contains the point $-x$ whenever it contains the point $x$, and if

$$f(-x) = f(x) \quad (1)$$

for each $x$ in the domain of $f$. Similarly, $f$ is an **odd function** if its domain contains $-x$ whenever it contains $x$, and if

$$f(-x) = -f(x) \quad (2)$$

for each $x$ in the domain of $f$. Examples of even functions are $1, x^2, \cos nx, |x|$, and $x^{2n}$. The functions $x, x^3, \sin nx$, and $x^{2n+1}$ are examples of odd functions. Note that according to Eq. (2), $f(0)$ must be zero if $f$ is an odd function whose domain contains the origin. Most functions are neither even nor odd, for instance, $e^x$. Only one function, $f$ identically zero, is both even and odd.

Elementary properties of even and odd functions include the following:

1. The sum (difference) and product (quotient) of two even functions are even.
2. The sum (difference) of two odd functions is odd; the product (quotient) of two odd functions is even.

3. The sum (difference) of an odd function and an even function is neither even nor odd; the product (quotient) of two such functions is odd.

The proofs of all these assertions are simple and follow directly from the definitions. For example, if both $f_1$ and $f_2$ are odd, and if $g(x) = f_1(x) + f_2(x)$, then
\[
g(-x) = f_1(-x) + f_2(-x) = -f_1(x) - f_2(x) \\
= -[f_1(x) + f_2(x)] = -g(x),
\]
so $f_1 + f_2$ is an odd function also. Similarly, if $h(x) = f_1(x)f_2(x)$, then
\[
h(-x) = f_1(-x)f_2(-x) = [-f_1(x)][-f_2(x)] = f_1(x)f_2(x) = h(x),
\]
so that $f_1f_2$ is even.

Also of importance are the following two integral properties of even and odd functions:

4. If $f$ is an even function, then
\[
\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx.
\]

5. If $f$ is an odd function, then
\[
\int_{-L}^{L} f(x)dx = 0.
\]

These properties are intuitively clear from the interpretation of an integral in terms of area under a curve, and they also follow immediately from the definitions. For example, if $f$ is even, then
\[
\int_{-L}^{L} f(x)dx = \int_{-L}^{0} f(x)dx + \int_{0}^{L} f(x)dx.
\]
Letting $x = -s$ in the first term on the right side and using Eq. (1), we obtain
The proof of the corresponding property for odd functions is similar.

Even and odd functions are particularly important in applications of Fourier series since their Fourier series have special forms, which occur frequently in physical problems.

**Cosine Series.** Suppose that $f$ and $f'$ are piecewise continuous on $-L \leq x < L$ and that $f$ is an even periodic function with period $2L$. Then it follows from properties 1 and 3 that $f(x) \cos\left(\frac{n\pi x}{L}\right)$ is even and $f(x) \sin\left(\frac{n\pi x}{L}\right)$ is odd. As a consequence of Eqs. (5) and (6), the Fourier coefficients of $f$ are then given by

\[
a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx, \quad n = 0, 1, 2, \ldots
\]

\[
b_n = 0, \quad n = 1, 2, \ldots
\]

Thus $f$ has the Fourier series

\[f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right).\]

In other words, the Fourier series of any even function consists only of the even trigonometric functions $\cos\left(\frac{n\pi x}{L}\right)$ and the constant term; it is natural to call such a series a **Fourier cosine series**. From a computational point of view, observe that only the coefficients $a_n$ for $n = 0, 1, 2, \ldots$ need to be calculated from the integral formula (7). Each of the $b_n$ for $n = 1, 2, \ldots$ is automatically zero for any even function and so does not need to be calculated by integration.

**Sine Series.** Suppose that $f$ and $f'$ are piecewise continuous on $-L \leq x < L$ and that $f$ is an odd periodic function of period $2L$. Then it follows from properties 2 and 3 that $f(x) \cos\left(\frac{n\pi x}{L}\right)$ is odd and $f(x) \sin\left(\frac{n\pi x}{L}\right)$ is even. In this case the Fourier coefficients of $f$ are

\[
a_n = 0, \quad n = 0, 1, 2, \ldots
\]

\[
b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx, \quad n = 1, 2, \ldots
\]

and the Fourier series for $f$ is of the form

\[f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).\]

Thus the Fourier series for any odd function consists only of the odd trigonometric functions $\sin\left(\frac{n\pi x}{L}\right)$; such a series is called a **Fourier sine series**. Again observe that only half of the coefficients need to be calculated by integration, since each $a_n$ for $n = 0, 1, 2, \ldots$ is zero for any odd function.

**Example** Let $f(x) = x$, $-L < x < L$ and let $f(-L) = f(L) = 0$. Let $f$ be defined elsewhere...
so that it is periodic of period 2L (see Figure 10.4.2). The function defined in this manner is known as a sawtooth wave. Find the Fourier series for this function.

**Figure 10.4.2** Sawtooth wave.

Since \( f \) is an odd function, its Fourier coefficients are, according to Eq. (8),

\[
a_n = 0, \quad n = 0, 1, 2, \ldots
\]

\[
b_n = \frac{2}{L} \int_0^L x \sin \frac{n\pi x}{L} \, dx
\]

\[
= \frac{2}{L} \left( \frac{L}{n\pi} \right)^2 \left[ \sin \frac{n\pi x}{L} - \frac{n\pi x}{L} \cos \frac{n\pi x}{L} \right]_0^L
\]

\[
= \frac{2L}{n\pi} \left( -1 \right)^{n+1}, \quad n = 1, 2, \ldots
\]

Hence the Fourier series for \( f \), the sawtooth wave, is

\[
f(x) = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{L}.
\]

(9)

Observe that the periodic function \( f \) is discontinuous at the points \( \pm L, \pm 3L \), as shown in Figure 10.4.2. At these points the series (9) converges to the mean value of the left and right limits, namely zero. The partial sum of the series (9) for \( n = 9 \) is shown in Figure 10.4.3. The Gibbs phenomenon (mentioned in Section 10.3) again occurs near the points of discontinuity.

**Figure 10.4.3** A partial sum in the Fourier series, Eq. (9), for the sawtooth wave.
Note that in this example \( f(-L) = f(L) = 0 \), as well as \( f(0) = 0 \). This is required if the function \( f \) is to be both odd and periodic with period \( 2L \). When we speak of constructing a sine series for a function defined on \( 0 \leq x \leq L \), it is understood that, if necessary, we must first redefine the function to be zero at \( x = 0 \) and \( x = L \).

It is worthwhile to observe that the triangular wave function (Example 1 of Section 10.2) and the sawtooth wave function just considered are identical on the interval \( 0 \leq x < L \). Therefore, their Fourier series converge to the same function, \( f(x) = x \), on this interval. Thus, if it is required to represent the function \( f(x) = x \) on \( 0 \leq x < L \) by a Fourier series, it is possible to do this by either a cosine series or a sine series. In the former case \( f \) is extended as an even function into the interval \(-L < x < 0\) and elsewhere periodically (the triangular wave). In the latter case \( f \) is extended into \(-L < x < 0\) as an odd function and elsewhere periodically (the sawtooth wave). If \( f \) is extended in any other way, the resulting Fourier series will still converge to \( x \) in \( 0 \leq x < L \) but will involve both sine and cosine terms.

In solving problems in differential equations it is often useful to expand in a Fourier series of period \( 2L \) a function \( f \) originally defined only on the interval \([0, L]\). As indicated previously for the function \( f(x) = x \), several alternatives are available. Explicitly, we can

1. Define a function \( g \) of period \( 2L \) so that

\[
g(x) = \begin{cases} 
  f(x), & 0 \leq x \leq L, \\
  f(-x), & -L < x < 0. 
\end{cases} \tag{10}
\]

The function \( g \) is thus the even periodic extension of \( f \). Its Fourier series, which is a cosine series, represents \( f \) on \([0, L]\).

2. Define a function \( h \) of period \( 2L \) so that

\[
h(x) = \begin{cases} 
  f(x), & 0 < x < L, \\
  0, & x = 0, L, \\
  -f(-x), & -L < x < 0. 
\end{cases} \tag{11}
\]

The function \( h \) is thus the odd periodic extension of \( f \). Its Fourier series, which is a sine series, also represents \( f \) on \((0, L)\).

3. Define a function \( k \) of period \( 2L \) so that

\[
k(x) = f(x), \quad 0 \leq x \leq L, \tag{12}
\]
and let \( k(x) \) be defined for \((-L, 0)\) in any way consistent with the conditions of Theorem 10.3.1. Sometimes it is convenient to define \( k(x) \) to be zero for \(-L < x < 0\). The Fourier series for \( k \), which involves both sine and cosine terms, also represents \( f \) on \([0, L]\), regardless of the manner in which \( k(x) \) is defined in \((-L, 0)\). Thus there are infinitely many such series, all of which converge to \( f(x) \) in the original interval.

Usually, the form of the expansion to be used will be dictated (or at least suggested) by the purpose for which it is needed. However, if there is a choice as to the kind of Fourier series to be used, the selection can sometimes be based on the rapidity of convergence. For example, the cosine series for the triangular wave [Eq. (20) of Section 10.2] converges more rapidly than the sine series for the sawtooth wave [Eq. (9) in this section], although both converge to the same function for \( 0 \leq x \leq L \). This is because the triangular wave is a smoother function than the sawtooth wave and is therefore easier to approximate. In general, the more continuous derivatives possessed by a function over the entire interval \(-\infty < x < \infty\), the faster its Fourier series will converge. See Problem 18 of Section 10.3.

**Example 2**

Suppose that

\[
 f(x) = \begin{cases} 
 1 - x, & 0 < x \leq 1, \\
 0, & 1 < x \leq 2. 
\end{cases}
\]  

(13)

As indicated previously, we can represent \( f \) either by a cosine series or by a sine series. Sketch the graph of the sum of each of these series for \(-6 \leq x \leq 6\).

In this example \( L = 2 \), so the cosine series for \( f \) converges to the even periodic extension of \( f \) of period 4, whose graph is sketched in Figure 10.4.4.

![Figure 10.4.4](image)

**Figure 10.4.4** Even periodic extension of \( f(x) \) given by Eq. (13).

Similarly, the sine series for \( f \) converges to the odd periodic extension of \( f \) of period 4. The graph of this function is shown in Figure 10.4.5.

![Figure 10.4.5](image)
PROBLEMS

In each of Problems 1 through 6 determine whether the given function is even, odd, or neither.

1. \( x^3 - 2x \)

2. \( x^3 - 2x + 1 \)

3. \( \tan 2x \)

4. \( \sec x \)

5. \( |x|^3 \)

6. \( e^{-x} \)

In each of Problems 7 through 12 a function \( f \) is given on an interval of length \( L \). In each case sketch the graphs of the even and odd extensions of \( f \) of period \( 2L \).

7. \[
   f(x) = \begin{cases} 
   x, & 0 \leq x < 2, \\
   1, & 2 \leq x < 3 
   \end{cases}
\]

8. \[
   f(x) = \begin{cases} 
   0, & 0 \leq x < 1, \\
   x - 1, & 1 \leq x < 2 
   \end{cases}
\]

Figure 10.4.5 Odd periodic extension of \( f(x) \) given by Eq. (13).
9. \( f(x) = 2 - x, \ 0 < x < 2 \)

10. \( f(x) = x - 3, \ 0 < x < 4 \)

11. \[
   f(x) = \begin{cases} 
   0, & 0 \leq x < 1, \\
   1, & 1 \leq x < 2 
   \end{cases}
\]

12. \( f(x) = 4 - x^2, \ 0 < x < 1 \)

13. Prove that any function can be expressed as the sum of two other functions, one of which is even and the other odd. That is, for any function \( f \), whose domain contains \(-x\) whenever it contains \( x \), show that there are an even function \( g \) and an odd function \( h \) such that \( f(x) = g(x) + h(x) \).

   *Hint:* What can you say about \( f(x) + f(-x) \)?

14. Find the coefficients in the cosine and sine series described in Example 2.

In each of Problems 15 through 22 find the required Fourier series for the given function, and sketch the graph of the function to which the series converges over three periods.

15. \[
   f(x) = \begin{cases} 
   1, & 0 < x < 1, \\
   0, & 1 < x < 2; 
   \end{cases}
\]
   
   cosine series, period 4

   Compare with Example 1 and Problem 5 of Section 10.3.

16. \[
   f(x) = \begin{cases} 
   x, & 0 \leq x < 1, \\
   1, & 1 \leq x < 2; 
   \end{cases}
\]
   
   sine series, period 4

   *Just Ask!*
17. \( f(x) = 1, \ 0 \leq x \leq \pi; \) cosine series, period \( 2\pi \)

18. \( f(x) = 1, \ 0 < x < \pi; \) sine series, period \( 2\pi \)

19. 
\[
\begin{aligned}
f(x) &= 0, \quad 0 < x < \pi, \\
&= 1, \quad \pi < x < 2\pi, \quad \text{sine series, period} \ 6\pi \\
&= 2, \quad 2\pi < x < 3\pi;
\end{aligned}
\]

20. \( f(x) = x, \ 0 \leq x < 1; \) series of period 1

21. \( f(x) = L - x, \ 0 \leq x \leq L; \) cosine series, period \( 2L \) Compare with Example 1 of Section 10.2.

Just Ask!

22. \( f(x) = L - x, \ 0 < x < L; \) sine series, period \( 2L \)

In each of Problems 23 through 26:

(a) Find the required Fourier series for the given function.

(b) Sketch the graph of the function to which the series converges for three periods.

(c) Plot one or more partial sums of the series.

23. \( f(x) = \begin{cases} 
x, & 0 < x < \pi, \\
0, & \pi < x < 2\pi; \end{cases} \) cosine series, period \( 4\pi \)

24. \( f(x) = -x, \ -\pi < x < 0; \) sine series, period \( 2\pi \)
25. \( f(x) = 2 - x^2, \ 0 < x < 2; \) \( \text{sine series, period 4} \)

26. \( f(x) = x^2 - 2x, \ 0 < x < 4; \) \( \text{cosine series, period 8} \)

In each of Problems 27 through 30 a function is given on an interval \( 0 < x < L. \)

(a) Sketch the graphs of the even extension \( g(x) \) and the odd extension \( h(x) \) of the given function of period \( 2L \) over three periods.

(b) Find the Fourier cosine and sine series for the given function.

(c) Plot a few partial sums of each series.

(d) For each series investigate the dependence on \( n \) of the maximum error on \( [0, L]. \)

27. \( f(x) = 3 - x, \ 0 < x < 3 \)

28. \( f(x) = \begin{cases} 
  x, & 0 < x < 1, \\
  0, & 1 < x < 2 
\end{cases} \)

29. \( f(x) = \left(4x^2 - 4x - 3\right)/4, \ 0 < x < 2 \)

30. \( f(x) = x^3 - 5x^2 + 5x + 1, \ 0 < x < 3 \)

31. Prove that if \( f \) is an odd function, then

\[ \int_{-L}^{L} f(x) \, dx = 0. \]
32. Prove properties 2 and 3 of even and odd functions, as stated in the text.

33. Prove that the derivative of an even function is odd and that the derivative of an odd function is even.

34. Let \( F(x) = \int_0^x f(t) \, dt \). Show that if \( f \) is even, then \( F \) is odd, and that if \( f \) is odd, then \( F \) is even.

35. From the Fourier series for the square wave in Example 1 of Section 10.3, show that

\[
\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.
\]

This relation between \( \pi \) and the odd positive integers was discovered by Leibniz in 1674.

36. From the Fourier series for the triangular wave (Example 1 of Section 10.2), show that

\[
\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.
\]

37. Assume that \( f \) has a Fourier sine series

\[
f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad 0 \leq x \leq L.
\]

(a) Show formally that

\[
\frac{2}{L} \int_0^L [f(x)]^2 \, dx = \sum_{n=1}^{\infty} b_n^2.
\]

Compare this result (Parseval's equation) with that of Problem 17 in Section 10.3. What is the corresponding result if \( f \) has a cosine series?

(b) Apply the result of part (a) to the series for the sawtooth wave given in Eq. (9), and thereby show that
This relation was discovered by Euler about 1735.

More Specialized Fourier Series. Let \( f \) be a function originally defined on \( 0 \leq x \leq L \) and satisfying there the continuity conditions of Theorem 10.3.1. In this section we have shown that it is possible to represent \( f \) by either a sine series or a cosine series by constructing odd or even periodic extensions of \( f \), respectively. Problems 38 through 40 concern some other, more specialized Fourier series that converge to the given function \( f \) on \((0, L)\).

38. Let \( f \) be extended into \((L, 2L]\) in an arbitrary manner. Then extend the resulting function into \((-2L, 0)\) as an odd function and elsewhere as a periodic function of period \(4L\) (see Figure 10.4.6). Show that this function has a Fourier sine series in terms of the functions \( \sin(n\pi x/2L) \) \( n = 1, 2, 3, \ldots \); that is,

\[
f(x) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{2L} \right),
\]

where

\[
b_n = \frac{1}{L} \int_{0}^{2L} f(x) \sin \left( \frac{n\pi x}{2L} \right) dx.
\]

This series converges to the original function on \((0, L)\).

Figure 10.4.6 Graph of the function in Problem 38.

39. Let \( f \) first be extended into \((L, 2L)\) so that it is symmetric about \( x = L \); that is, so as to satisfy \( f(2L - x) = f(x) \) for \( 0 \leq x \leq L \). Let the resulting function be extended into \((-2L, 0)\) as an odd function and elsewhere (see Figure 10.4.7) as a periodic function of period \(4L\). Show that this function has a Fourier series in terms of the functions \( \sin(\pi x/2L), \sin(3\pi x/2L), \sin(5\pi x/2L), \ldots \); that is,

\[
f(x) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{(2n-1)\pi x}{2L} \right),
\]
where

\[ b_n = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{(2n-1)\pi x}{2L} \right) dx. \]

This series converges to the original function on \((0, L)\).

**Figure 10.4.7** Graph of the function in Problem 39.

**40.** How should \( f \), originally defined on \([0, L]\), be extended so as to obtain a Fourier series involving only the functions \( \cos(\pi x/2L) \), \( \cos(3\pi x/2L) \), \( \cos(5\pi x/2L) \)…? Refer to Problems 38 and 39. If \( f(x) = x \) for \( 0 \leq x \leq L \), sketch the function to which the Fourier series converges for \(-4L \leq x \leq 4L\).
10.5 Separation of Variables; Heat Conduction in a Rod

The basic partial differential equations of heat conduction, wave propagation, and potential theory that we discuss in this chapter are associated with three distinct types of physical phenomena: diffusive processes, oscillatory processes, and time-independent or steady processes. Consequently, they are of fundamental importance in many branches of physics. They are also of considerable significance from a mathematical point of view. The partial differential equations whose theory is best developed and whose applications are most significant and varied are the linear equations of second order. All such equations can be classified into one of three categories: The heat conduction equation, the wave equation, and the potential equation, respectively, are prototypes of each of these categories. Thus a study of these three equations yields much information about more general second order linear partial differential equations.

During the last two centuries several methods have been developed for solving partial differential equations. The method of separation of variables is the oldest systematic method, having been used by D'Alembert, Daniel Bernoulli, and Euler about 1750 in their investigations of waves and vibrations. It has been considerably refined and generalized in the meantime, and it remains a method of great importance and frequent use today. To show how the method of separation of variables works we consider first a basic problem of heat conduction in a solid body. The mathematical study of heat conduction originated about 1800, and it continues to command the attention of modern scientists. For example, analysis of the dissipation and transfer of heat away from its sources in high-speed machinery is frequently an important technological problem.

Let us now consider a heat conduction problem for a straight bar of uniform cross section and homogeneous material. Let the $x$-axis be chosen to lie along the axis of the bar, and let $x = 0$ and $x = L$ denote the ends of the bar (see Figure 10.5.1). Suppose further that the sides of the bar are perfectly insulated so that no heat passes through them. We also assume that the cross-sectional dimensions are so small that the temperature $u$ can be considered constant on any given cross section. Then $u$ is a function only of the axial coordinate $x$ and the time $t$.

![Figure 10.5.1 A heat-conducting solid bar.](image)

The variation of temperature in the bar is governed by a partial differential equation whose derivation appears in Appendix A at the end of this chapter. The equation is called the heat
The conduction equation and has the form

\[ \alpha^2 u_{xx} = u_t, \quad 0 < x < L, \quad t > 0, \]  

where \( \alpha^2 \) is a constant known as the thermal diffusivity. The parameter \( \alpha^2 \) depends only on the material from which the bar is made and is defined by

\[ \alpha^2 = \frac{\kappa}{\rho s}, \]  

where \( \kappa \) is the thermal conductivity, \( \rho \) is the density, and \( s \) is the specific heat of the material in the bar. The units of \( \alpha^2 \) are \((\text{length})^2/\text{time}\). Typical values of \( \alpha^2 \) are given in Table 10.5.1.

**Table 10.5.1** Values of the Thermal Diffusivity for Some Common Materials

<table>
<thead>
<tr>
<th>Material</th>
<th>( \alpha^2 ) (cm(^2)/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Silver</td>
<td>1.71</td>
</tr>
<tr>
<td>Copper</td>
<td>1.14</td>
</tr>
<tr>
<td>Aluminum</td>
<td>0.86</td>
</tr>
<tr>
<td>Cast iron</td>
<td>0.12</td>
</tr>
<tr>
<td>Granite</td>
<td>0.011</td>
</tr>
<tr>
<td>Brick</td>
<td>0.0038</td>
</tr>
<tr>
<td>Water</td>
<td>0.00144</td>
</tr>
</tbody>
</table>

In addition, we assume that the initial temperature distribution in the bar is given; thus
where \( f \) is a given function. Finally, we assume that the ends of the bar are held at fixed temperatures: the temperature \( T_1 \) at \( x = 0 \) and the temperature \( T_2 \) at \( x = L \). However, it turns out that we need only consider the case where \( T_1 = T_2 = 0 \). We show in Section 10.6 how to reduce the more general problem to this special case. Thus in this section we will assume that \( u \) is always zero when \( x = 0 \) or \( x = L \):

\[
 u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0.
\]  

(4)

The fundamental problem of heat conduction is to find \( u(x, t) \) that satisfies the differential equation (1) for \( 0 < x < L \) and for \( t > 0 \), the initial condition (3) when \( t = 0 \), and the boundary conditions (4) at \( x = 0 \) and \( x = L \).

The problem described by Eqs. (1), (3), and (4) is an initial value problem in the time variable \( t \); an initial condition is given and the differential equation governs what happens later. However, with respect to the space variable \( x \), the problem is a boundary value problem; boundary conditions are imposed at each end of the bar and the differential equation describes the evolution of the temperature in the interval between them. Alternatively, we can consider the problem as a boundary value problem in the \( xt \)-plane (see Figure 10.5.2). The solution \( u(x, t) \) of Eq. (1) is sought in the semi-infinite strip \( 0 < x < L, \ t > 0 \), subject to the requirement that \( u(x, t) \) must assume a prescribed value at each point on the boundary of this strip.

![Figure 10.5.2 Boundary value problem for the heat conduction equation.](image)

The heat conduction problem (1), (3), (4) is linear since \( u \) appears only to the first power throughout. The differential equation and boundary conditions are also homogeneous. This suggests that we might approach the problem by seeking solutions of the differential equation and boundary conditions, and then superposing them to satisfy the initial condition. The remainder of this section describes how this plan can be implemented.

One solution of the differential equation (1) that satisfies the boundary conditions (4) is the
function \( u(x, t) = 0 \), but this solution does not satisfy the initial condition (3) except in the trivial case in which \( f(x) \) is also zero. Thus our goal is to find other, nonzero solutions of the differential equation and boundary conditions. To find the needed solutions we start by making a basic assumption about the form of the solutions that has far-reaching, and perhaps unforeseen, consequences. The assumption is that \( u(x, t) \) is a product of two other functions, one depending only on \( x \) and the other depending only on \( t \); thus

\[
u(x, t) = X(x)T(t).
\]

Substituting from Eq. (5) for \( u \) in the differential equation (1) yields

\[
\alpha^2 X'' T = X T',
\]

where primes refer to ordinary differentiation with respect to the independent variable, whether \( x \) or \( t \). Equation (6) is equivalent to

\[
\frac{X''}{X} = \frac{1}{\alpha^2} \frac{T'}{T}.
\]

in which the variables are separated; that is, the left side depends only on \( x \) and the right side only on \( t \). For Eq. (7) to be valid for \( 0 < x < L \), \( t > 0 \), it is necessary that both sides of Eq. (7) be equal to the same constant. Otherwise, if one independent variable (say \( x \)) were kept fixed and the other were allowed to vary, one side (the left in this case) of Eq. (7) would remain unchanged while the other varied, thus violating the equality. If we call this separation constant \( \lambda \), then Eq. (7) becomes

\[
\frac{X''}{X} = \frac{1}{\alpha^2} \frac{T'}{T} = -\lambda.
\]

Hence we obtain the following two ordinary differential equations for \( X(x) \) and \( T(t) \):

\[
X'' + \lambda X = 0, \tag{9}
\]

\[
T' + \alpha^2 \lambda T = 0. \tag{10}
\]

We denote the separation constant by \(-\lambda\) (rather than \(\lambda\)) because it turns out that it must be negative, and it is convenient to exhibit the minus sign explicitly.

The assumption (5) has led to the replacement of the partial differential equation (1) by the two ordinary differential equations (9) and (10). Each of these equations can be readily solved for any value of \( \lambda \). The product of two solutions of Eq. (9) and (10), respectively, provides a solution of the partial differential equation (1). However, we are interested only in those solutions of Eq. (1) that...
also satisfy the boundary conditions (4). As we now show, this severely restricts the possible values of \( \lambda \).

Substituting for \( u(x, t) \) from Eq. (5) in the boundary condition at \( x = 0 \), we obtain

\[
 u(0, t) = X(0)T(t) = 0. \tag{11}
\]

If Eq. (11) is satisfied by choosing \( T(t) \) to be zero for all \( t \), then \( u(x, t) \) is zero for all \( x \) and \( t \), and we have already rejected this possibility. Therefore Eq. (11) must be satisfied by requiring that

\[
 X(0) = 0. \tag{12}
\]

Similarly, the boundary condition at \( x = L \) requires that

\[
 X(L) = 0. \tag{13}
\]

We now want to consider Eq. (9) subject to the boundary conditions (12) and (13). This is an eigenvalue problem and, in fact, is the same problem that we discussed in detail at the end of Section 10.1; see especially the paragraph following Eq. (29) in that section. The only difference is that the dependent variable there was called \( y \) rather than \( X \). If we refer to the results obtained earlier [Eq. (31) of Section 10.1], the only nontrivial solutions of Eqs. (9), (12), and (13) are the eigenfunctions

\[
 X_n(x) = \sin(n\pi x/L), \quad n = 1, 2, 3, \ldots \tag{14}
\]

associated with the eigenvalues

\[
 \lambda_n = n^2\pi^2/L^2, \quad n = 1, 2, 3, \ldots \tag{15}
\]

Turning now to Eq. (10) for \( I(t) \) and substituting \( -n^2\pi^2\alpha^2/L^2 \) for \( \lambda \), we have

\[
 T' + \left\{ -n^2\pi^2\alpha^2/L^2 \right\} T = 0. \tag{16}
\]

Thus \( I(t) \) is proportional to \( \exp \left\{ -n^2\pi^2\alpha^2 t/L^2 \right\} \). Hence, multiplying solutions of Eqs. (9) and (10) together, and neglecting arbitrary constants of proportionality, we conclude that the functions

\[
 u_n(x, t) = e^{-n^2\pi^2\alpha^2 t/L^2} \sin(n\pi x/L), \quad n = 1, 2, 3, \ldots \tag{17}
\]

satisfy the partial differential equation (1) and the boundary conditions (4) for each positive integer value of \( n \).
The functions \( u_n \) are sometimes called fundamental solutions of the heat conduction problem (1), (3), and (4).

It remains only to satisfy the initial condition (3),

\[
    u(x, 0) = f(x), \quad 0 \leq x \leq L. \tag{18}
\]

Recall that we have often solved initial value problems by forming linear combinations of a set of fundamental solutions and then choosing the coefficients to satisfy the initial conditions. The analogous step in the present problem is to form a linear combination of the functions (17) and then to choose the coefficients to satisfy Eq. (18). The main difference from earlier problems is that there are infinitely many functions (17), so a general linear combination of them is an infinite series. Thus we assume that

\[
    u(x, t) = \sum_{n=1}^{\infty} c_n u_n(x, t) = \sum_{n=1}^{\infty} c_n e^{-n^2 \pi^2 \alpha^2 t/L^2} \sin \frac{n \pi x}{L}, \tag{19}
\]

where the coefficients \( c_n \) are as yet undetermined. The individual terms in the series (19) satisfy the differential equation (1) and boundary conditions (4). We will assume that the infinite series of Eq. (19) converges and also satisfies Eqs. (1) and (4). To satisfy the initial condition (3) we must have

\[
    u(x, 0) = \sum_{n=1}^{\infty} c_n \sin \frac{n \pi x}{L} = f(x). \tag{20}
\]

In other words, we need to choose the coefficients \( c_n \) so that the series of sine functions in Eq. (20) converges to the initial temperature distribution \( f(x) \) for \( 0 \leq x \leq L \). The series in Eq. (20) is just the Fourier sine series for \( f \); according to Eq. (8) of Section 10.4 its coefficients are given by

\[
    c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n \pi x}{L} \, dx. \tag{21}
\]

Hence the solution of the heat conduction problem of Eqs. (1), (3), and (4) is given by the series in Eq. (19) with the coefficients computed from Eq. (21).

**Example 1**

Find the temperature \( u(x, t) \) at any time in a metal rod 50 cm long, insulated on the sides, which initially has a uniform temperature of 20°C throughout and whose ends are maintained at 0°C for all \( t > 0 \).

The temperature in the rod satisfies the heat conduction problem (1), (3), (4) with \( L = 50 \) and \( f(x) = 20 \) for \( 0 < x < 50 \). Thus, from Eq. (19), the solution is

\[
    u(x, t) = \sum_{n=1}^{\infty} c_n e^{-n^2 \pi^2 \alpha^2 t/2500} \sin \frac{n \pi x}{50}, \tag{22}
\]

where, from Eq. (21),
Finally, by substituting for $c_n$ in Eq. (22), we obtain

$$u(x, t) = \frac{80}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} e^{-n^2 \alpha^2 t / 2500} \sin \frac{n \pi x}{50}.$$  

(24)

The expression (24) for the temperature is moderately complicated, but the negative exponential factor in each term of the series causes the series to converge quite rapidly, except for small values of $t$ or $\alpha^2$. Therefore accurate results can usually be obtained by using only a few terms of the series.

In order to display quantitative results, let us measure $t$ in seconds; then $\alpha^2$ has the units of $\text{cm}^2/\text{sec}$. If we choose $\alpha^2 = 1$ for convenience, this corresponds to a rod of a material whose thermal properties are somewhere between copper and aluminum. The behavior of the solution can be seen from the graphs in Figures 10.5.3 through 10.5.5. In Figure 10.5.3 we show the temperature distribution in the bar at several different times. Observe that the temperature diminishes steadily as heat in the bar is lost through the end points. The way in which the temperature decays at a given point in the bar is indicated in Figure 10.5.4, where temperature is plotted against time for a few selected points in the bar. Finally, Figure 10.5.5 is a three-dimensional plot of $u$ versus both $x$ and $t$. Observe that we obtain the graphs in Figures 10.5.3 and 10.5.4 by intersecting the surface in Figure 10.5.5 by planes on which either $t$ or $x$ is constant. The slight waviness in Figure 10.5.5 at $t = 0$ results from using only a finite number of terms in the series for $u(x, t)$ and from the slow convergence of the series for $t = 0$.

**Figure 10.5.3** Temperature distributions at several times for the heat conduction problem of Example 1.
A problem with possible practical implications is to determine the time $\tau$ at which the entire bar has cooled to a specified temperature. For example, when is the temperature in the entire bar no greater than 1°C? Because of the symmetry of the initial temperature distribution and the boundary conditions, the warmest point in the bar is always the center. Thus $\tau$ is found by solving $u(25, t) = 1$ for $t$. Using one term in the series expansion (24), we obtain

$$\tau = \frac{2500}{\pi^2} \ln(80/\pi) \approx 820 \text{ sec.}$$

**PROBLEMS**
In each of Problems 1 through 6 determine whether the method of separation of variables can be used to replace the given partial differential equation by a pair of ordinary differential equations. If so, find the equations.

1. \( xu_{xx} + u_t = 0 \)

2. \( tu_{xx} + xu_t = 0 \)

3. \( u_{xx} + u_{xt} + u_t = 0 \)

4. \( [p(x) u_x]_x - r(x) u_{tt} = 0 \)

5. \( u_{xx} + (x + y) u_{yy} = 0 \)

6. \( u_{xx} + u_{yy} + xu = 0 \)

7. Find the solution of the heat conduction problem

\[
100u_{xx} = u_t, \quad 0 < x < 1, \quad t > 0; \\
u(0, t) = 0, \quad u(1, t) = 0, \quad t > 0; \\
u(x, 0) = \sin 2\pi x - \sin 5\pi x, \quad 0 \leq x \leq 1.
\]

8. Find the solution of the heat conduction problem

\[
u_{xx} = 4u_t, \quad 0 < x < 2, \quad t > 0; \\
u(0, t) = 0, \quad u(2, t) = 0, \quad t > 0; \\
u(x, 0) = 2\sin(\pi x / 2) - \sin \pi x + 4\sin 2\pi x, \quad 0 \leq x \leq 2.
\]

Consider the conduction of heat in a rod 40 cm in length whose ends are maintained at 0°C for all \( t > 0 \). In each of
Problems 9 through 12 find an expression for the temperature \( u(x,t) \) if the initial temperature distribution in the rod is the given function. Suppose that \( \alpha^2 = 1 \).

9. \( u(x,0) = 50, \quad 0 < x < 40 \)

10. \[
u(x,0) = \begin{cases} 
  x, & 0 \leq x < 20, \\
  40 - x, & 20 \leq x \leq 40
\end{cases}
\]

11. \[
u(x,0) = \begin{cases} 
  0, & 0 \leq x < 10, \\
  50, & 10 \leq x \leq 30, \\
  0, & 30 < x \leq 40
\end{cases}
\]

12. \( u(x,0) = x, \quad 0 < x < 40 \)

Just Ask!

13. Consider again the rod in Problem 9. For \( t = 5 \) and \( x = 20 \) determine how many terms are needed to find the solution correct to three decimal places. A reasonable way to do this is to find \( n \) so that including one more term does not change the first three decimal places of \( u(20, 5) \). Repeat for \( t = 20 \) and \( t = 80 \). Form a conclusion about the speed of convergence of the series for \( u(x,t) \).

14. For the rod in Problem 9:

(a) Plot \( u \) versus \( x \) for \( t = 5, 10, 20, 40, 100, \) and 200. Put all of the graphs on the same set of axes and thereby obtain a picture of the way in which the temperature distribution changes with time.

(b) Plot \( u \) versus \( t \) for \( x = 5, 10, 15, \) and 20.

(c) Draw a three-dimensional plot of \( u \) versus \( x \) and \( t \).

(d) How long does it take for the entire rod to cool off to a temperature of no more than 1°C?
15. Follow the instructions in Problem 14 for the rod in Problem 10.

16. Follow the instructions in Problem 14 for the rod in Problem 11.

17. For the rod in Problem 12:
   (a) Plot $u$ versus $x$ for $t = 5, 10, 20, 40, 100,$ and $200$.
   (b) For each value of $t$ used in part (a) estimate the value of $x$ for which the temperature is greatest. Plot these values versus $t$ to see how the location of the warmest point in the rod changes with time.
   (c) Plot $u$ versus $t$ for $x = 10, 20,$ and $30$.
   (d) Draw a three-dimensional plot of $u$ versus $x$ and $t$.
   (e) How long does it take for the entire rod to cool off to a temperature of no more than $1^\circ C$?

18. Let a metallic rod $20$ cm long be heated to a uniform temperature of $100^\circ C$. Suppose that at $t = 0$ the ends of the bar are plunged into an ice bath at $0^\circ C$, and thereafter maintained at this temperature, but that no heat is allowed to escape through the lateral surface. Find an expression for the temperature at any point in the bar at any later time. Determine the temperature at the center of the bar at time $t = 30$ sec if the bar is made of (a) silver, (b) aluminum, or (c) cast iron.

19. For the rod of Problem 18 find the time that will elapse before the center of the bar cools to a temperature of $5^\circ C$ if the bar is made of (a) silver, (b) aluminum, or (c) cast iron.

20. In solving differential equations the computations can almost always be simplified by the use of dimensionless variables. Show that if the dimensionless variable $\xi = x/L$ is introduced, the heat conduction equation becomes

$$\frac{\partial^2 u}{\partial \xi^2} = \frac{L^2}{\alpha^2} \frac{\partial u}{\partial t}, \quad 0 < \xi < 1, \quad t > 0.$$
variable \( \tau = \left( \frac{\alpha^2}{L^2} \right) t \). Then show that the heat conduction equation reduces to

\[
\frac{\partial^2 u}{\partial \xi^2} t = \frac{\partial u}{\partial \tau}, \quad 0 < \xi < 1, \quad \tau > 0.
\]

21. Consider the equation

\[
au_{xx} - bu_t + cu = 0,
\]

where \( a, b, \) and \( c \) are constants.

(a) Let \( u(x, t) = e^{\delta t} w(x, t) \) where \( \delta \) is constant, and find the corresponding partial differential equation for \( w \).

(b) If \( b \neq 0 \), show that \( \delta \) can be chosen so that the partial differential equation found in part (a) has no term in \( w \). Thus, by a change of dependent variable, it is possible to reduce Eq. (i) to the heat conduction equation.

22. The heat conduction equation in two space dimensions is

\[
\alpha^2 (u_{xx} + u_{yy}) = u_t.
\]

Assuming that \( u(x, y, t) = X(x)Y(y)T(t) \), find ordinary differential equations that are satisfied by \( X(x), Y(y), \) and \( T(t) \).

23. The heat conduction equation in two space dimensions may be expressed in terms of polar coordinates as

\[
\alpha^2 \left[ u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} \right] = u_t.
\]

Assuming that \( u(r, \theta, t) = R(r) \Theta(\theta)T(t) \), find ordinary differential equations that are satisfied by \( R(r), \Theta(\theta), \) and \( T(t) \).
10.6 Other Heat Conduction Problems

In Section 10.5 we considered the problem consisting of the heat conduction equation

\[ \alpha^2 u_{xx} = u_t, \quad 0 < x < L, \quad t > 0, \quad (1) \]

the boundary conditions

\[ u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0, \quad (2) \]

and the initial condition

\[ u(x, 0) = f(x), \quad 0 \leq x \leq L. \quad (3) \]

We found the solution to be

\[ u(x, t) = \sum_{n=1}^{\infty} c_n e^{-n^2 \pi^2 \alpha^2 t/L^2} \sin \frac{n\pi x}{L}, \quad (4) \]

where the coefficients \( c_n \) are the same as in the series

\[ f(x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L}. \quad (5) \]

The series in Eq. (5) is just the Fourier sine series for \( f \); according to Section 10.4 its coefficients are given by

\[ c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx. \quad (6) \]

Hence the solution of the heat conduction problem, Eqs. (1) to (3), is given by the series in Eq. (4) with the coefficients computed from Eq. (6).

We emphasize that at this stage the solution (4) must be regarded as a formal solution; that is, we obtained it without rigorous justification of the limiting processes involved. Such a justification is beyond the scope of this book. However, once the series (4) has been obtained, it is possible to show that in \( 0 < x < L, \ t > 0 \) it converges to a continuous function, that the derivatives \( u_{xx} \) and \( u_t \) can be computed by differentiating the series (4) term by term, and that the heat conduction equation (1) is indeed satisfied. The argument relies heavily on the fact that each term of the series (4) contains a negative exponential factor, and this results in relatively rapid convergence of the series. A further argument establishes that the function \( u \) given by Eq. (4) also satisfies the
boundary and initial conditions; this completes the justification of the formal solution.

It is interesting to note that although $f$ satisfies the conditions of the Fourier convergence theorem (Theorem 10.3.1), it may have points of discontinuity. In this case the initial temperature distribution $u(x,0) = f(x)$ is discontinuous at one or more points. Nevertheless, the solution $u(x,t)$ is continuous for arbitrarily small values of $t > 0$. This illustrates the fact that heat conduction is a diffusive process that instantly smooths out any discontinuities that may be present in the initial temperature distribution. Finally, since $f$ is bounded, it follows from Eq. (6) that the coefficients $c_n$ are also bounded. Consequently, the presence of the negative exponential factor in each term of the series (4) guarantees that

$$\lim_{t \to \infty} u(x,t) = 0$$  \hspace{1cm} (7)$$

for all $x$ regardless of the initial condition. This is in accord with the result expected from physical intuition.

We now consider two other problems of one-dimensional heat conduction that can be handled by the method developed in Section 10.5.

**Nonhomogeneous Boundary Conditions.** Suppose now that one end of the bar is held at a constant temperature $T_1$ and the other is maintained at a constant temperature $T_2$. Then the boundary conditions are

$$u(0,t) = T_1, \quad u(L,t) = T_2, \quad t > 0.$$  \hspace{1cm} (8)$$

The differential equation (1) and the initial condition (3) remain unchanged.

This problem is only slightly more difficult, because of the nonhomogeneous boundary conditions, than the one in Section 10.5. We can solve it by reducing it to a problem having homogeneous boundary conditions, which can then be solved as in Section 10.5. The technique for doing this is suggested by the following physical argument.

After a long time—that is, as $t \to \infty$ —we anticipate that a steady temperature distribution $v(x)$ will be reached, which is independent of the time $t$ and the initial conditions. Since $v(x)$ must satisfy the equation of heat conduction (1), we have

$$v''(x) = 0, \quad 0 < x < L.$$  \hspace{1cm} (9)$$

Hence the steady-state temperature distribution is a linear function of $x$. Further, $v(x)$ must satisfy the boundary conditions

$$v(0) = T_1, \quad v(L) = T_2.$$  \hspace{1cm} (10)$$
which are valid even as $t \to \infty$. The solution of Eq. (9) satisfying Eqs. (10) is

$$v(x) = (T_2 - T_1) \frac{x}{L} + T_1. \quad (11)$$

Returning to the original problem, Eqs. (1), (3), and (8), we will try to express $u(x, t)$ as the sum of the steady-state temperature distribution $v(x)$ and another (transient) temperature distribution $w(x)$; thus we write

$$u(x, t) = v(x) + w(x, t). \quad (12)$$

Since $v(x)$ is given by Eq. (11), the problem will be solved, provided that we can determine $w(x, t)$. The boundary value problem for $w(x, t)$ is found by substituting the expression in Eq. (12) for $u(x, t)$ in Eqs. (1), (3), and (8).

From Eq. (1) we have

$$\alpha^2 (v + w)_{xx} = (v + w)_t,$$

it follows that

$$\alpha^2 w_{xx} = w_t, \quad (13)$$

since $v_{xx} = 0$ and $v_t = 0$. Similarly, from Eqs. (12), (8), and (10),

$$w(0, t) = u(0, t) - v(0) = T_1 - T_1 = 0,$$
$$w(L, t) = u(L, t) - v(L) = T_2 - T_2 = 0. \quad (14)$$

Finally, from Eqs. (12) and (3),

$$w(x, 0) = u(x, 0) - v(x) = f(x) - v(x), \quad (15)$$

where $v(x)$ is given by Eq. (11). Thus the transient part of the solution to the original problem is found by solving the problem consisting of Eqs. (13), (14), and (15). This latter problem is precisely the one solved in Section 10.5, provided that $f(x) - v(x)$ is now regarded as the initial temperature distribution. Hence

$$u(x, t) = (T_2 - T_1) \frac{x}{L} + T_1 + \sum_{n=1}^{\infty} c_n e^{-n^2 \pi^2 \alpha^2 t/L^2} \sin \frac{n \pi x}{L}, \quad (16)$$

where

$$c_n = \frac{2}{L} \int_0^L \left[ f(x) - (T_2 - T_1) \frac{x}{L} - T_1 \right] \sin \frac{n \pi x}{L} \, dx. \quad (17)$$
This is another case in which a more difficult problem is solved by reducing it to a simpler problem that has already been solved. The technique of reducing a problem with nonhomogeneous boundary conditions to one with homogeneous boundary conditions by subtracting the steady-state solution has wide application.

**Example 1**

Consider the heat conduction problem

\[
\begin{align*}
    u_{xx} &= u_t, \quad 0 < x < 30, \quad t > 0, \\
    u(0, t) &= 20, \quad u(30, t) = 50, \quad t > 0, \\
    u(x, 0) &= 60 - 2x, \quad 0 < x < 30.
\end{align*}
\]

Find the steady-state temperature distribution and the boundary value problem that determines the transient distribution.

The steady-state temperature satisfies \(v''(x) = 0\) and the boundary conditions \(v(0) = 20\) and \(v(30) = 50\). Thus \(v(x) = 20 + x\). The transient distribution \(w(x, t)\) satisfies the heat conduction equation

\[
w_{xx} = w_t,
\]

the homogeneous boundary conditions

\[
w(0, t) = 0, \quad w(30, t) = 0,
\]

and the modified initial condition

\[
w(x, 0) = 60 - 2x - (20 + x) = 40 - 3x.
\]

Note that this problem is of the form (1), (2), (3) with \(f(x) = 40 - 3x\), \(a^2 = 1\), and \(L = 30\). Thus the solution is given by Eqs. (4) and (6).

Figure 10.6.1 shows a plot of the initial temperature distribution \(60 - 2x\), the final temperature distribution \(20 + x\), and the temperature at three intermediate times found by solving Eqs. (21) through (23). Note that the intermediate temperature satisfies the boundary conditions (19) for any \(t > 0\). As \(t\) increases, the effect of the boundary conditions gradually moves from the ends of the bar toward its center.
Bar with Insulated Ends. A slightly different problem occurs if the ends of the bar are insulated so that there is no passage of heat through them. According to Eq. (2) in Appendix A, the rate of flow of heat across a cross section is proportional to the rate of change of temperature in the $x$ direction. Thus, in the case of no heat flow, the boundary conditions are

$$u_x(0, t) = 0, \quad u_x(L, t) = 0, \quad t > 0. \quad (24)$$

The problem posed by Eqs. (1), (3), and (24) can also be solved by the method of separation of variables. If we let

$$u(x, t) = X(x) T(t), \quad (25)$$

and substitute for $u$ in Eq. (1), then it follows, as in Section 10.5, that

$$\frac{X''}{X} = \frac{1}{\alpha^2} \frac{T'}{T} = -\lambda, \quad (26)$$

where $\lambda$ is a constant. Thus we obtain again the two ordinary differential equations

$$X'' + \lambda X = 0, \quad (27)$$

$$T' + \alpha^2 \lambda T = 0. \quad (28)$$

For any value of $\lambda$ a product of solutions of Eqs. (27) and (28) is a solution of the partial differential equation (1). However, we are interested only in those solutions that also satisfy the boundary conditions (24).
If we substitute for \( u(x, t) \) from Eq. (25) in the boundary condition at \( x = 0 \), we obtain \( X'(0)T(t) = 0 \). We cannot permit \( T(t) \) to be zero for all \( t \), since then \( u(x, t) \) would also be zero for all \( t \). Hence we must have

\[
X'(0) = 0. \tag{29}
\]

Proceeding in the same way with the boundary condition at \( x = L \), we find that

\[
X'(L) = 0. \tag{30}
\]

Thus we wish to solve Eq. (27) subject to the boundary conditions (29) and (30). It is possible to show that nontrivial solutions of this problem can exist only if \( \lambda \) is real. One way to show this is indicated in Problem 18; alternatively, we can appeal to a more general theory to be discussed in Section 11.2. We will assume that \( \lambda \) is real and will consider in turn the three cases \( \lambda < 0 \), \( \lambda = 0 \), and \( \lambda > 0 \).

If \( \lambda < 0 \), it is convenient to let \( \lambda = -\mu^2 \), where \( \mu \) is real and positive. Then Eq. (27) becomes

\[
X'' - \mu^2 X = 0,
\]

and its general solution is

\[
X(x) = k_1 \sinh \mu x + k_2 \cosh \mu x. \tag{31}
\]

In this case the boundary conditions can be satisfied only by choosing \( k_1 = k_2 = 0 \). Since this is unacceptable, it follows that \( \lambda \) cannot be negative; in other words, the problem (27), (29), (30) has no negative eigenvalues.

If \( \lambda = 0 \), then Eq. (27) is \( X'' = 0 \), and therefore

\[
X(x) = k_1 x + k_2. \tag{32}
\]

The boundary conditions (29) and (30) require that \( k_1 = 0 \) but do not determine \( k_2 \). Thus \( \lambda = 0 \) is an eigenvalue, corresponding to the eigenfunction \( X(x) = 1 \). For \( \lambda = 0 \) it follows from Eq. (28) that \( T(t) \) is also a constant, which can be combined with \( k_2 \). Hence, for \( \lambda = 0 \), we obtain the constant solution \( u(x, t) = k_2 \).

Finally, if \( \lambda > 0 \), let \( \lambda = \mu^2 \), where \( \mu \) is real and positive. Then Eq. (27) becomes \( X'' + \mu^2 X = 0 \), and consequently

\[
X(x) = k_1 \sin \mu x + k_2 \cos \mu x. \tag{33}
\]

The boundary condition (29) requires that \( k_1 = 0 \), and the boundary condition (30) requires that \( \mu = n\pi/L \) for \( n = 1, 2, 3, \ldots \) but leaves \( k_2 \) arbitrary. Thus the problem (27), (29), (30) has an infinite sequence of positive eigenvalues \( \lambda = n^2 \pi^2/L^2 \) with the corresponding eigenfunctions \( X(x) = \cos(n\pi x/L) \). For these values of \( \lambda \)
the solutions $\mathcal{T}(t)$ of Eq. (28) are proportional to $\exp\left\{ -n^2 \pi^2 \alpha^2 \frac{t}{L^2} \right\}$.

Combining all these results, we have the following fundamental solutions for the problem (1), (3), and (24):

\begin{align}
  u_0(x, t) &= 1, \\
  u_n(x, t) &= e^{-n^2 \pi^2 \alpha^2 \frac{t}{L^2}} \cos \frac{n \pi x}{L}, \quad n = 1, 2, \ldots, \tag{34}
\end{align}

where arbitrary constants of proportionality have been dropped. Each of these functions satisfies the differential equation (1) and the boundary conditions (24). Because both the differential equation and the boundary conditions are linear and homogeneous, any finite linear combination of the fundamental solutions satisfies them. We will assume that this is true for convergent infinite linear combinations of fundamental solutions as well. Thus, to satisfy the initial condition (3), we assume that $u(x, t)$ has the form

\begin{align}
  u(x, t) &= \frac{c_0}{2} u_0(x, t) + \sum_{n=1}^{\infty} c_n u_n(x, t) \\
  &= \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-n^2 \pi^2 \alpha^2 \frac{t}{L^2}} \cos \frac{n \pi x}{L}. \tag{35}
\end{align}

The coefficients $c_n$ are determined by the requirement that

\begin{align}
  u(x, 0) &= \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos \frac{n \pi x}{L} = f(x). \tag{36}
\end{align}

Thus the unknown coefficients in Eq. (35) must be the coefficients in the Fourier cosine series of period $2L$ for $f$. Hence

\begin{align}
  c_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{n \pi x}{L} \, dx, \quad n = 0, 1, 2, \ldots \tag{37}
\end{align}

With this choice of the coefficients $c_0, c_2, c_2, \ldots$, the series (35) provides the solution to the heat conduction problem for a rod with insulated ends, Eqs. (1), (3), and (24).

It is worth observing that the solution (35) can also be thought of as the sum of a steady-state temperature distribution (given by the constant $c_0/2$), which is independent of time $t$, and a transient distribution (given by the rest of the infinite series) that vanishes in the limit as $t$ approaches infinity. That the steady state is a constant is consistent with the expectation that the process of heat conduction will gradually smooth out the temperature distribution in the bar as long as no heat is allowed to escape to the outside. The physical interpretation of the term

\begin{align}
  \frac{c_0}{2} = \frac{1}{L} \int_0^L f(x) \, dx \tag{38}
\end{align}

is that it is the mean value of the original temperature distribution.
**Example 2**

Find the temperature \( u(x, t) \) in a metal rod of length 25 cm that is insulated on the ends as well as on the sides and whose initial temperature distribution is \( u(x, 0) = x \) for \( 0 < x < 25 \).

The temperature in the rod satisfies the heat conduction problem (1), (3), (24) with \( L = 25 \). Thus, from Eq. (35), the solution is

\[
 u(x, t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-n^2 \pi^2 \alpha^2 t / 625} \cos \frac{n \pi x}{25},
\]

where the coefficients are determined from Eq. (37). We have

\[
 c_0 = \frac{2}{25} \int_0^{25} x \, dx = 25
\]

and, for \( n \geq 1 \),

\[
 c_n = \frac{2}{25} \int_0^{25} x \cos \frac{n \pi x}{25} \, dx = \frac{50}{n \pi} (\cos n \pi - 1)/(n \pi)^2 = \begin{cases} -\frac{100}{n \pi^2}, & n \text{ odd;} \\ 0, & n \text{ even.} \end{cases}
\]

Thus

\[
 u(x, t) = \frac{25}{2} - \frac{100}{\pi^2} \sum_{n=1,3,5,...}^{\infty} \frac{1}{n^2} e^{-n^2 \pi^2 \alpha^2 t / 625} \cos(n \pi x / 25)
\]

is the solution of the given problem.

For \( \alpha^2 = 1 \), Figure 10.6.2 shows plots of the temperature distribution in the bar at several times. Again the convergence of the series is rapid so that only a relatively few terms are needed to generate the graphs.

**Figure 10.6.2** Temperature distributions at several times for the heat conduction
More General Problems. The method of separation of variables can also be used to solve heat conduction problems with other boundary conditions than those given by Eqs. (8) and Eqs. (24). For example, the left end of the bar might be held at a fixed temperature \( T \) while the other end is insulated. In this case the boundary conditions are

\[
\begin{align*}
  u(0, t) &= T, \quad u_x(L, t) = 0, \quad t > 0.
\end{align*}
\]

The first step in solving this problem is to reduce the given boundary conditions to homogeneous ones by subtracting the steady-state solution. The resulting problem is solved by essentially the same procedure as in the problems previously considered. However, the extension of the initial function \( f \) outside of the interval \([0, L]\) is somewhat different from that in any case considered so far (see Problem 15).

A more general type of boundary condition occurs when the rate of heat flow through the end of the bar is proportional to the temperature. It is shown in Appendix A that the boundary conditions in this case are of the form

\[
\begin{align*}
  u_x(0, t) - h_1 u(0, t) &= 0, \quad u_x(L, t) + h_2 u(L, t) = 0, \quad t > 0,
\end{align*}
\]

where \( h_1 \) and \( h_2 \) are nonnegative constants. If we apply the method of separation of variables to the problem consisting of Eqs. (1), (3), and (44), we find that \( X(x) \) must be a solution of

\[
\begin{align*}
  X'''' + \lambda X &= 0, \quad X'(0) - h_1 X(0) = 0, \quad X'(L) + h_2 X(L) = 0,
\end{align*}
\]

where \( \lambda \) is the separation constant. Once again it is possible to show that nontrivial solutions can exist only for certain nonnegative real values of \( \lambda \), the eigenvalues, but these values are not given by a simple formula (see Problem 20). It is also possible to show that the corresponding solutions of Eqs. (45), the eigenfunctions, satisfy an orthogonality relation and that one can satisfy the initial condition (3) by superposing solutions of Eqs. (45). However, the resulting series is not included in the discussion of this chapter. There is more general theory that covers such problems, and it is outlined in Chapter 11.

PROBLEMS

In each of Problems 1 through 8 find the steady-state solution of the heat conduction equation \( \alpha^2 u_{xx} = u_t \) that satisfies the given set of boundary conditions.
1. \( u(0, t) = 10, \quad u(50, t) = 40 \)

2. \( u(0, t) = 30, \quad u(40, t) = -20 \)

   Just Ask!

3. \( u_x(0, t) = 0, \quad u(L, t) = 0 \)

4. \( u_x(0, t) = 0, \quad u(L, t) = T \)

5. \( u(0, t) = 0, \quad u_x(L, t) = 0 \)

6. \( u(0, t) = T, \quad u_x(L, t) = 0 \)

7. \( u_x(0, t) - u(0, t) = 0, \quad u(L, t) = T \)

8. \( u(0, t) = T, \quad u_x(L, t) + u(L, t) = 0 \)

9. Let an aluminum rod of length 20 cm be initially at the uniform temperature of 25°C. Suppose that at time \( t = 0 \) the end \( x = 0 \) is cooled to 0°C while the end \( x = 20 \) is heated to 60°C, and both are thereafter maintained at those temperatures.

   (a) Find the temperature distribution in the rod at any time \( t \).

   (b) Plot the initial temperature distribution, the final (steady-state) temperature distribution, and the temperature distributions at two representative intermediate times on the same set of axes.

   (c) Plot \( u \) versus \( t \) for \( x = 5, 10, \) and 15.

   (d) Determine how much time must elapse before the temperature at \( x = 5 \) cm comes (and remains)
10. Let the ends of a copper rod 100 cm long be maintained at 0°C. Suppose that the center of the bar is heated to 100°C by an external heat source and that this situation is maintained until a steady state results. Find this steady-state temperature distribution.

(b) At a time $t = 0$ [after the steady state of part (a) has been reached], let the heat source be removed. At the same instant let the end $x = 0$ be placed in thermal contact with a reservoir at 20°C, while the other end remains at 0°C. Find the temperature as a function of position and time.

(c) Plot $u$ versus $x$ for several values of $t$. Also plot $u$ versus $t$ for several values of $x$.

(d) What limiting value does the temperature at the center of the rod approach after a long time? How much time must elapse before the center of the rod cools to within 1° of its limiting value?

11. Consider a rod of length 30 for which $\alpha^2 = 1$. Suppose the initial temperature distribution is given by $u(x, 0) = x(60 - x)/30$ and that the boundary conditions are $u(0, t) = 30$ and $u(30, t) = 0$.

(a) Find the temperature in the rod as a function of position and time.

(b) Plot $u$ versus $x$ for several values of $t$. Also plot $u$ versus $t$ for several values of $x$.

(c) Plot $u$ versus $t$ for $x = 12$. Observe that $u$ initially decreases, then increases for a while, and finally decreases to approach its steady-state value. Explain physically why this behavior occurs at this point.

12. Consider a uniform rod of length $L$ with an initial temperature given by $u(x, 0) = \sin(\pi x/L)$, $0 \leq x \leq L$. Assume that both ends of the bar are insulated.

(a) Find the temperature $u(x, t)$.

(b) What is the steady-state temperature as $t \to \infty$?
13. Consider a bar of length 40 cm whose initial temperature is given by \( u(x,0) = x(60-x)/30 \). Suppose that \( \alpha^2 = 1/4 \text{ cm}^2/\text{sec} \) and that both ends of the bar are insulated.

(a) Find the temperature \( u(x,t) \).

(b) Plot \( u \) versus \( x \) for several values of \( t \). Also plot \( u \) versus \( t \) for several values of \( x \).

(c) Determine the steady-state temperature in the bar.

(d) Determine how much time must elapse before the temperature at \( x = 40 \) comes within 1° of its steady-state value.

14. Consider a bar 30 cm long that is made of a material for which \( \alpha^2 = 1 \) and whose ends are insulated. Suppose that the initial temperature is zero except for the interval \( 5 < x < 10 \), where the initial temperature is 25°.

(a) Find the temperature \( u(x,t) \).

(b) Plot \( u \) versus \( x \) for several values of \( t \). Also plot \( u \) versus \( t \) for several values of \( x \).

(c) Plot \( u(4, t) \) and \( u(11, t) \) versus \( t \). Observe that the points \( x = 4 \) and \( x = 11 \) are symmetrically located with respect to the initial temperature pulse, yet their temperature plots are significantly different. Explain physically why this is so.

15. Consider a uniform bar of length \( L \) having an initial temperature distribution given by \( f(x), 0 \leq x \leq L \). Assume that the temperature at the end \( x = 0 \) is held at 0°C, while the end \( x = L \) is insulated so that no heat passes through it.
(a) Show that the fundamental solutions of the partial differential equation and boundary conditions are

\[ u_n(x, t) = e^{-(2n-1)^2 \pi^2 \alpha^2 t / 4L^2} \sin \left( \frac{(2n-1) \pi x}{2L} \right), \quad n = 1, 2, 3, \ldots \]

(b) Find a formal series expansion for the temperature \( u(x, t) \),

\[ u(x, t) = \sum_{n=1}^{\infty} c_n u_n(x, t), \]

that also satisfies the initial condition \( u(x, 0) = f(x) \).

\textit{Hint:} Even though the fundamental solutions involve only the odd sines, it is still possible to represent \( f \) by a Fourier series involving only these functions. See Problem 39 of Section 10.4.

16. In the bar of Problem 15 suppose that \( L = 30 \), \( \alpha^2 = 1 \), and the initial temperature distribution is \( f(x) = 30 - x \) for \( 0 < x < 30 \).

(a) Find the temperature \( u(x, t) \).

(b) Plot \( u \) versus \( x \) for several values of \( t \). Also plot \( u \) versus \( t \) for several values of \( x \).

(c) How does the location \( x_m \) of the warmest point in the bar change as \( t \) increases? Draw a graph of \( x_m \) versus \( t \).

(d) Plot the maximum temperature in the bar versus \( t \).

17. Suppose that the conditions are as in Problems 15 and 16 except that the boundary condition at \( x = 0 \) is \( u(0, t) = 40 \).

(a) Find the temperature \( u(x, t) \).

(b) Plot \( u \) versus \( x \) for several values of \( t \). Also plot \( u \) versus \( t \) for several values of \( x \).

(c) Compare the plots you obtained in this problem with those from Problem 16. Explain how the change in the boundary condition at \( x = 0 \) causes the observed differences in the behavior of the temperature in the bar.
18. Consider the problem

\[ X'' + \lambda X = 0, \quad X'(0) = 0, \quad X'(L) = 0. \] \hspace{1cm} (i)

Let \( \lambda = \mu^2 \), where \( \mu = \nu + i\sigma \) with \( \nu \) and \( \sigma \) real. Show that if \( \sigma \neq 0 \), then the only solution of Eqs. (i) is the trivial solution \( X(x) = 0 \).

*Hint:* Use an argument similar to that in Problem 23 of Section 10.1.

**Just Ask!**

19. The right end of a bar of length \( a \) with thermal conductivity \( \kappa_1 \) and cross-sectional area \( A_1 \) is joined to the left end of a bar of thermal conductivity \( \kappa_2 \) and cross-sectional area \( A_2 \). The composite bar has a total length \( L \). Suppose that the end \( x = 0 \) is held at temperature zero, while the end \( x = L \) is held at temperature \( T \). Find the steady-state temperature in the composite bar, assuming that the temperature and rate of heat flow are continuous at \( x = a \).

*Hint:* See Eq. (2) of Appendix A.

20. Consider the problem

\[
\begin{align*}
\alpha^2 u_{xx} &= u_t, & 0 < x < L, \quad t > 0; \\
u(0, t) &= 0, & u_x(L, t) + \gamma u(L, t) = 0, & t > 0; \\
u(x, 0) &= f(x), & 0 \leq x \leq L. 
\end{align*}
\] \hspace{1cm} (i)

(a) Let \( u(x, t) = X(x) T(t) \) and show that

\[ X'' + \lambda X = 0, \quad X(0) = 0, \quad X'(L) + \gamma X(L) = 0, \] \hspace{1cm} (ii)

and

\[ T' + \lambda \alpha^2 T = 0, \]

where \( \lambda \) is the separation constant.

(b) Assume that \( \lambda \) is real, and show that problem (ii) has no nontrivial solutions if \( \lambda \leq 0 \).
(c) If \( \lambda > 0 \), let \( \lambda = \mu^2 \) with \( \mu > 0 \). Show that problem (ii) has nontrivial solutions only if \( \mu \) is a solution of the equation

\[
\mu \cos \mu L + \gamma \sin \mu L = 0.
\]  

(iii)

(d) Rewrite Eq. (iii) as \( \tan \mu L = -\mu / \gamma \). Then, by drawing the graphs of \( y = \tan \mu L \) and \( y = -\mu L / \gamma L \) for \( \mu > 0 \) on the same set of axes, show that Eq. (iii) is satisfied by infinitely many positive values of \( \mu \); denote these by \( \mu_1, \mu_2, \ldots, \mu_n, \ldots \) ordered in increasing size.

(e) Determine the set of fundamental solutions \( u_n(x, t) \) corresponding to the values \( \mu_n \) found in part (d).

An External Heat Source. Consider the heat conduction problem in a bar that is in thermal contact with an external heat source or sink. Then the modified heat conduction equation is

\[
u_t = \alpha^2 u_{xx} + s(x),
\]  

(i)

where the term \( s(x) \) describes the effect of the external agency; \( s(x) \) is positive for a source and negative for a sink. Suppose that the boundary conditions are

\[
u(0, t) = T_1, \quad \nu(L, t) = T_2
\]  

(ii)

and the initial condition is

\[
u(x, 0) = f(x).
\]  

(iii)

Problems 21 through 23 deal with this kind of problem.

21. Write \( \nu(x, t) = v(x) + w(x, t) \), where \( v \) and \( w \) are the steady state and transient parts of the solution, respectively. State the boundary value problems that \( v(x) \) and \( w(x, t) \), respectively, satisfy. Observe that the problem for \( w \) is the fundamental heat conduction problem discussed in Section 10.5, with a modified initial temperature distribution.

22. Suppose that \( \alpha^2 = 1 \) and \( s(x) = k \), a constant, in Eq. (i). Find \( v(x) \).
(b) Assume that \( T_1 = 0, T_2 = 0, L = 20, \ k = 1/5 \), and that \( f(x) = 0 \) for \( 0 < x < L \). Determine \( w(x, t) \). Then plot \( u(x, t) \) versus \( x \) for several values of \( t \); on the same axes also plot the steady-state part of the solution \( v(x) \).

23.

(a) Let \( \alpha^2 = 1 \) and \( s(x) = k\alpha/L \), where \( k \) is a constant, in Eq. (i). Find \( v(x) \).

(b) Assume that \( T_1 = 10, T_2 = 30, L = 20, k = 1/2 \), and that \( f(x) = 0 \) for \( 0 < x < L \). Determine \( w(x, t) \). Then plot \( u(x, t) \) versus \( x \) for several values of \( t \); on the same axes also plot the steady-state part of the solution \( v(x) \).
10.7 The Wave Equation: Vibrations of an Elastic String

A second partial differential equation that occurs frequently in applied mathematics is the wave\(^*\) equation. Some form of this equation, or a generalization of it, almost inevitably arises in any mathematical analysis of phenomena involving the propagation of waves in a continuous medium. For example, the studies of acoustic waves, water waves, electromagnetic waves, and seismic waves are all based on this equation.

Perhaps the easiest situation to visualize occurs in the investigation of mechanical vibrations. Suppose that an elastic string of length \(L\) is tightly stretched between two supports at the same horizontal level, so that the \(x\)-axis lies along the string (see Figure 10.7.1). The elastic string may be thought of as a violin string, a guy wire, or possibly an electric power line. Suppose that the string is set in motion (by plucking, for example) so that it vibrates in a vertical plane, and let \(u(x, t)\) denote the vertical displacement experienced by the string at the point \(x\) at time \(t\). If damping effects, such as air resistance, are neglected, and if the amplitude of the motion is not too large, then \(u(x, t)\) satisfies the partial differential equation

\[
a^2 u_{xx} = u_{tt}
\]

in the domain \(0 < x < L, t > 0\). Equation (1) is known as the one-dimensional wave equation and is derived in Appendix B at the end of the chapter. The constant coefficient \(a^2\) appearing in Eq. (1) is given by

\[
a^2 = \frac{T}{\rho},
\]

where \(T\) is the tension (force) in the string, and \(\rho\) is the mass per unit length of the string material. It follows that \(a\) has the units of length/time—that is, of velocity. In Problem 14 it is shown that \(a\) is the velocity of propagation of waves along the string.

To describe the motion of the string completely it is necessary also to specify suitable initial and boundary conditions for the displacement \(u(x, t)\). The ends are assumed to remain fixed, and therefore the boundary conditions are
Since the differential equation (1) is of second order with respect to \( t \), it is plausible to prescribe two initial conditions. These are the initial position of the string,

\[
    u(0, t) = 0, \quad u(L, t) = 0, \quad t \geq 0. \tag{3}
\]

and its initial velocity,

\[
    u(x, 0) = f(x), \quad 0 \leq x \leq L, \tag{4}
\]

\[
    u_t(x, 0) = g(x), \quad 0 \leq x \leq L, \tag{5}
\]

where \( f \) and \( g \) are given functions. In order for Eqs. (3), (4), and (5) to be consistent it is also necessary to require that

\[
    f(0) = f(L) = 0, \quad g(0) = g(L) = 0. \tag{6}
\]

The mathematical problem then is to determine the solution of the wave equation (1) that also satisfies the boundary conditions (3) and the initial conditions (4) and (5). Like the heat conduction problem of Sections 5 and 6, this problem is an initial value problem in the time variable \( t \) and a boundary value problem in the space variable \( x \). Alternatively, it can be considered as a boundary value problem in the semi-infinite strip \( 0 < x < L, \quad t > 0 \) of the \( x t \)-plane (see Figure 10.7.2). One condition is imposed at each point on the semi-infinite sides, and two are imposed at each point on the finite base.

**Figure 10.7.2** Boundary value problem for the wave equation.

It is important to realize that Eq. (1) governs a large number of other wave problems besides the transverse vibrations of an elastic string. For example, it is only necessary to interpret the function...
and the constant $a$ appropriately to have problems dealing with water waves in an ocean, acoustic or electromagnetic waves in the atmosphere, or elastic waves in a solid body. If more than one space dimension is significant, then Eq. (1) must be slightly generalized. The two-dimensional wave equation is

$$a^2(u_{xx} + u_{yy}) = u_{tt}. \quad (7)$$

This equation would arise, for example, if we considered the motion of a thin elastic sheet, such as a drumhead. Similarly, in three dimensions the wave equation is

$$a^2(u_{xx} + u_{yy} + u_{zz}) = u_{tt}. \quad (8)$$

In connection with the latter two equations, the boundary and initial conditions must also be suitably generalized.

We now solve some typical boundary value problems involving the one-dimensional wave equation.

**Elastic String with Nonzero Initial Displacement.** First suppose that the string is disturbed from its equilibrium position and then released at time $t = 0$ with zero velocity to vibrate freely. Then the vertical displacement $u(x, t)$ must satisfy the wave equation (1),

$$a^2 u_{xx} = u_{tt}, \quad 0 < x < L, \quad t > 0;$$

the boundary conditions (3),

$$u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0;$$

and the initial conditions

$$u(x, 0) = f(x), \quad u_t(x, 0) = 0, \quad 0 \leq x \leq L, \quad (9)$$

where $f$ is a given function describing the configuration of the string at $t = 0$.

The method of separation of variables can be used to obtain the solution of Eqs. (1), (3), and (9). Assuming that

$$u(x, t) = X(x)T(t) \quad (10)$$

and substituting for $u$ in Eq. (1), we obtain

$$\frac{X'''}{X} = \frac{1}{a^2} \frac{T''}{T} = -\lambda,$$  

where $\lambda$ is a separation constant. Thus we find that $X(x)$ and $T(t)$ satisfy the ordinary differential equations
Further, by substituting from Eq. (10) for \( u(x, t) \) in the boundary conditions (3), we find that \( X(x) \) must satisfy the boundary conditions

\[
X(0) = 0, \quad X(L) = 0. \tag{14}
\]

Finally, by substituting from Eq. (10) into the second of the initial conditions (9), we also find that \( T(t) \) must satisfy the initial condition

\[
T'(0) = 0. \tag{15}
\]

Our next task is to determine \( X(x) \), \( T(t) \), and \( \lambda \) by solving Eq. (12) subject to the boundary conditions (14) and Eq. (13) subject to the initial condition (15).

The problem of solving the differential equation (12) subject to the boundary conditions (14) is precisely the same problem that arose in Section 10.5 in connection with the heat conduction equation. Thus we can use the results obtained there and at the end of Section 10.1: The problem (12), (14) has nontrivial solutions if and only if \( \lambda \) is an eigenvalue,

\[
\lambda = n^2 \pi^2 / L^2, \quad n = 1, 2, \ldots \tag{16}
\]

and \( X(x) \) is proportional to the corresponding eigenfunction \( \sin \left( \frac{n\pi x}{L} \right) \).

Using the values of \( \lambda \) given by Eq. (16) in Eq. (13), we obtain

\[
T'' + \frac{n^2 \pi^2}{L^2} T = 0. \tag{17}
\]

Therefore

\[
T(t) = k_1 \cos \frac{n\pi at}{L} + k_2 \sin \frac{n\pi at}{L}, \tag{18}
\]

where \( k_1 \) and \( k_2 \) are arbitrary constants. The initial condition (15) requires that \( k_2 = 0 \), so \( T(t) \) must be proportional to \( \cos \left( \frac{n\pi at}{L} \right) \).

Thus the functions
satisfy the partial differential equation (1), the boundary conditions (3), and the second initial condition (9). These functions are the fundamental solutions for the given problem.

To satisfy the remaining (nonhomogeneous) initial condition (9) we will consider a superposition of the fundamental solutions (19) with properly chosen coefficients. Thus we assume that $u(x, t)$ has the form

$$
    u(x, t) = \sum_{n=1}^{\infty} c_n u_n(x, t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} \cos \frac{n\pi at}{L},
$$  \hspace{1cm} (20)

where the constants $c_n$ remain to be chosen. The initial condition $u(x, 0) = f(x)$ requires that

$$
    u(x, 0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} = f(x).
$$  \hspace{1cm} (21)

Consequently, the coefficients $c_n$ must be the coefficients in the Fourier sine series of period $2L$ for $f$; hence

$$
    c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx, \quad n = 1, 2, \ldots
$$  \hspace{1cm} (22)

Thus the formal solution of the problem of Eqs. (1), (3), and (9) is given by Eq. (20) with the coefficients calculated from Eq. (22).

For a fixed value of $n$ the expression $\sin \left( \frac{n\pi x}{L} \right)$ in Eq. (19) is periodic in time $t$ with the period $2L/na$; it therefore represents a vibratory motion of the string having this period, or having the frequency $n \pi a/L$. The quantities $\lambda_n = n\pi a/L$ for $n = 1, 2, \ldots$ are the natural frequencies of the string—that is, the frequencies at which the string will freely vibrate. The factor $\sin \left( \frac{n\pi x}{L} \right)$ represents the displacement pattern occurring in the string when it is executing vibrations of the given frequency. Each displacement pattern is called a natural mode of vibration and is periodic in the space variable $x$; the spatial period $2L/n$ is called the wavelength of the mode of frequency $n \pi a/L$. Thus the eigenvalues $n^2 \pi^2 / L^2$ of the problem (12), (14) are proportional to the squares of the natural frequencies, and the eigenfunctions $\sin \left( \frac{n\pi x}{L} \right)$ give the natural modes. The first three natural modes are sketched in Figure 10.7.3. The total motion of the string, given by the function $u(x, t)$ of Eq. (20), is thus a combination of the natural modes of vibration and is also a periodic function of time with period $2L/a$. 

![Graphs of natural modes](image-url)
**Example 1**

Consider a vibrating string of length $L = 30$ that satisfies the wave equation

$$4u_{xx} = u_{tt}, \quad 0 < x < 30, \quad t > 0. \quad (23)$$

Assume that the ends of the string are fixed and that the string is set in motion with no initial velocity from the initial position

$$u(x, 0) = f(x) = \begin{cases} \frac{x}{10}, & 0 \leq x \leq 10, \\ \frac{30-x}{20}, & 10 < x \leq 30. \end{cases} \quad (24)$$

Find the displacement $u(x, t)$ of the string and describe its motion through one period.

The solution is given by Eq. (20) with $a = 2$ and $L = 30$; that is,

$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{30} \cos \frac{2n\pi t}{30}. \quad (25)$$

where $c_n$ is calculated from Eq. (22). Substituting from Eq. (24) into Eq. (22), we obtain

$$c_n = \frac{2}{30} \int_{0}^{10} \frac{x}{10} \sin \frac{n\pi x}{30} \, dx + \frac{2}{30} \int_{10}^{30} \frac{30-x}{20} \sin \frac{n\pi x}{30} \, dx. \quad (26)$$

By evaluating the integrals in Eq. (26), we find that

$$c_n = \frac{9}{n^2 \pi^2} \sin \frac{n\pi}{3}, \quad n = 1, 2, \ldots \quad (27)$$

The solution (25), (27) gives the displacement of the string at any point $x$ at any time $t$. The motion is periodic in time with period 30, so it is sufficient to analyze the solution for $0 \leq t \leq 30$.

The best way to visualize the solution is by a computer animation showing the dynamic behavior of the vibrating string. Here we indicate the motion of the string in Figures 10.7.4, 10.7.5, and 10.7.6. Plots of $u$ versus $x$ for $t = 0, 4, 7.5, 11, and 15$ are shown in Figure 10.7.4. Observe that the maximum initial displacement is positive and occurs at $x = 10$, while at $t = 15$, a half-period later, the maximum displacement is negative and occurs at $x = 20$. The string then retraces its motion and returns to its original configuration at $t = 30$. Figure 10.7.5 shows the behavior of the points $x = 10, 15, and 20$ by plots of $u$ versus $t$ for these fixed values of $x$. The plots confirm that the motion is indeed periodic with period 30. Observe also that each interior point on the string is motionless for one-
third of each period. Figure 10.7.6 shows a three-dimensional plot of $u$ versus both $x$ and $t$, from which the overall nature of the solution is apparent. Of course, the curves in Figures 10.7.4 and 10.7.5 lie on the surface shown in Figure 10.7.6.

**Figure 10.7.4** Plots of $u$ versus $x$ for fixed values of $t$ for the string in Example 1.

**Figure 10.7.5** Plots of $u$ versus $t$ for fixed values of $x$ for the string in Example 1.
Justification of the Solution. As in the heat conduction problem considered earlier, Eq. (20) with the coefficients $c_n$ given by Eq. (22) is only a formal solution of Eqs. (1), (3), and (9). To ascertain whether Eq. (20) actually represents the solution of the given problem requires some further investigation. As in the heat conduction problem, it is tempting to try to show this directly by substituting Eq. (20) for $u(x, t)$ in Eqs. (1), (3), and (9). However, upon formally computing $u_{xx}$, for example, we obtain

$$u_{xx}(x, t) = -\sum_{n=1}^{\infty} c_n \left( \frac{n \pi}{L} \right)^2 \sin \left( \frac{n \pi x}{L} \right) \cos \left( \frac{n \pi at}{L} \right);$$

due to the presence of the $n^2$ factor in the numerator, this series may not converge. This would not necessarily mean that the series (20) for $u(x, t)$ is incorrect, but only that the series (20) cannot be used to calculate $u_{xx}$ and $u_{tt}$. A basic difference between solutions of the wave equation and those of the heat conduction equation is that the latter contain negative exponential terms that approach zero very rapidly with increasing $n$, which ensures the convergence of the series solution and its derivatives. In contrast, series solutions of the wave equation contain only oscillatory terms that do not decay with increasing $n$.

However, there is an alternative way to establish the validity of Eq. (20) indirectly. At the same time, we will gain additional information about the structure of the solution. First we will show that Eq. (20) is equivalent to

$$u(x, t) = \frac{1}{2} \left[ h(x - at) + h(x + at) \right], \quad (28)$$

where $h$ is the function obtained by extending the initial data $f$ into $(-L, 0)$ as an odd function, and to other values of $x$ as a periodic function of period $2L$. That is,
To establish Eq. (28) note that $h$ has the Fourier series

$$h(x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L},$$

(30)

where $c_n$ is given by Eq. (22). Then, using the trigonometric identities for the sine of a sum or difference, we obtain

$$h(x - at) = \sum_{n=1}^{\infty} c_n \left( \sin \frac{n\pi x}{L} \cos \frac{n\pi at}{L} - \cos \frac{n\pi x}{L} \sin \frac{n\pi at}{L} \right),$$

$$h(x + at) = \sum_{n=1}^{\infty} c_n \left( \sin \frac{n\pi x}{L} \cos \frac{n\pi at}{L} + \cos \frac{n\pi x}{L} \sin \frac{n\pi at}{L} \right),$$

and Eq. (28) follows immediately upon adding the last two equations. From Eq. (28) we see that $u(x, t)$ is continuous for $0 < x < L$, $t > 0$, provided that $h$ is continuous on the interval $(-\infty, \infty)$. This requires that $f$ be continuous on the original interval $[0, L]$. Similarly, $u$ is twice continuously differentiable with respect to either variable in $0 < x < L$, $t > 0$, provided that $h$ is twice continuously differentiable on $(-\infty, \infty)$. This requires that $f'$ and $f''$ be continuous on $[0, L]$. Furthermore, since $h''$ is the odd extension of $f''$, we must also have $f''(0) = f''(L) = 0$. However, since $h'$ is the even extension of $f'$, no further conditions are required on $f'$. Provided that these conditions are met, $u_{xx}$ and $u_{tt}$ can be computed from Eq. (28), and it is an elementary exercise to show that these derivatives satisfy the wave equation. Some of the details of the argument just indicated are given in Problems 19 and 20.

If some of the continuity requirements stated in the last paragraph are not met, then $u$ is not differentiable at some points in the semi-infinite strip $0 < x < L$, $t > 0$, and thus is a solution of the wave equation only in a somewhat restricted sense. An important physical consequence of this observation is that if there are any discontinuities present in the initial data $f$, then they will be preserved in the solution $u(x, t)$ for all time. In contrast, in heat conduction problems, initial discontinuities are instantly smoothed out (Section 10.6). Suppose that the initial displacement $f$ has a jump discontinuity at $x = x_0$, $0 \leq x_0 \leq L$. Since $h$ is a periodic extension of $f$, the same discontinuity is present in $h(\xi)$ at $\xi = x_0 + 2nL$ and at $\xi = -x_0 + 2nL$, where $n$ is any integer. Thus $h(x - at)$ is discontinuous when $x - at = x_0 + 2nL$, or when $x - at = -x_0 + 2nL$. For a fixed $x$ in $[0, L]$ the discontinuity that was originally at $x_0$ will reappear in $h(x - at)$ at the times $t = (x - x_0 - 2nL)/a$. Similarly, $h(x + at)$ is discontinuous at the point $x$ at the times $t = (-x + x_0 + 2mL)/a$, where $m$ is any integer. If we refer to Eq. (28), it then follows that the solution $u(x, t)$ is also discontinuous at the given point $x$ at these times. Since the physical problem is posed for $t > 0$, only those values of $m$ and $n$ that yield positive values of $t$ are of interest.

**General Problem for the Elastic String.** Let us modify the problem considered previously by supposing that the string is set in motion from its equilibrium position with a given velocity. Then
the vertical displacement $u(x, t)$ must satisfy the wave equation (1),

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < L, \quad t > 0;$$

the boundary conditions (3),

$$u(0, t) = 0, \quad u(L, t) = 0, \quad t \geq 0;$$

and the initial conditions

$$u(x, 0) = 0, \quad u_t(x, 0) = g(x), \quad 0 \leq x \leq L, \quad (31)$$

where $g(x)$ is the initial velocity at the point $x$ of the string.

The solution of this new problem can be obtained by following the procedure described above for the problem (1), (3), and (9). Upon separating variables, we find that the problem for $X(x)$ is exactly the same as before. Thus, once again, $\lambda = n^2 \pi^2 / L^2$ and $X(x)$ is proportional to $\sin (n\pi x / L)$. The differential equation for $T(t)$ is again Eq. (17), but the associated initial condition is now

$$T(0) = 0, \quad (32)$$

corresponding to the first of the initial conditions (31). The general solution of Eq. (17) is given by Eq. (18), but now the initial condition (32) requires that $k_1 = 0$. Therefore $T(t)$ is now proportional to $\sin (n\pi at / L)$ and the fundamental solutions for the problem (1), (3), and (31) are

$$u_n(x, t) = \sin \frac{n\pi x}{L} \sin \frac{n\pi at}{L}, \quad n = 1, 2, 3, \ldots \quad (33)$$

Each of the functions $u_n(x, t)$ satisfies the wave equation (1), the boundary conditions (3), and the first of the initial conditions (31). The main consequence of using the initial conditions (31) rather than (9) is that the time-dependent factor in $u_n(x, t)$ involves a sine rather than a cosine.

To satisfy the remaining (nonhomogeneous) initial condition we assume that $u(x, t)$ can be expressed as a linear combination of the fundamental solutions (33); that is,

$$u(x, t) = \sum_{n=1}^{\infty} k_n u_n(x, t) = \sum_{n=1}^{\infty} k_n \sin \frac{n\pi x}{L} \sin \frac{n\pi at}{L}. \quad (34)$$

To determine the values of the coefficients $k_n$, we differentiate Eq. (34) with respect to $t$, set $t = 0$, and use the second initial condition (31); this gives the equation

$$u_t(x, 0) = \sum_{n=1}^{\infty} \frac{n\pi a}{L} k_n \sin \frac{n\pi x}{L} = g(x). \quad (35)$$

Hence the quantities $(n\pi a / L)k_n$ are the coefficients in the Fourier sine series of period $2L$ for $g$. Therefore
Thus Eq. (34), with the coefficients given by Eq. (36), constitutes a formal solution to the problem of Eqs. (1), (3), and (31). The validity of this formal solution can be established by arguments similar to those previously outlined for the solution of Eqs. (1), (3), and (9).

Finally, we turn to the problem consisting of the wave equation (1), the boundary conditions (3), and the general initial conditions (4), (5):

\[
u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L,\]

where \(f(x)\) and \(g(x)\) are the given initial position and velocity, respectively, of the string. Although this problem can be solved by separating variables, as in the cases discussed previously, it is important to note that it can also be solved simply by adding together the two solutions that we obtained above. To show that this is true, let \(v(x, t)\) be the solution of the problem (1), (3), and (9), and let \(w(x, t)\) be the solution of the problem (1), (3), and (31). Thus \(v(x, t)\) is given by Eqs. (20) and (22), and \(w(x, t)\) is given by Eqs. (34) and (36). Now let \(u(x, t) = v(x, t) + w(x, t)\); what problem does \(u(x, t)\) satisfy? First, observe that

\[
a^2 u_{xx} - u_{tt} = \left(a^2 v_{xx} - v_{tt}\right) + \left(a^2 w_{xx} - w_{tt}\right) = 0 + 0 = 0, \quad (38)
\]

so \(u(x, t)\) satisfies the wave equation (1). Next, we have

\[
u(0,t) = v(0,t) + w(0,t) = 0 + 0 = 0, \quad u(L,t) = v(L,t) + w(L,t) = 0 + 0 = 0, \quad (39)
\]

so \(u(x, t)\) also satisfies the boundary conditions (3). Finally, we have

\[
u(x,0) = v(x,0) + w(x,0) = f(x) + 0 = f(x) \quad (40)
\]

and

\[
u_t(x,0) = v_t(x,0) + w_t(x,0) = 0 + g(x) = g(x) \quad (41)
\]

Thus \(u(x, t)\) satisfies the general initial conditions (37).

We can restate the result we have just obtained in the following way. To solve the wave equation with the general initial conditions (37) you can solve instead the somewhat simpler problems with the initial conditions (9) and (31), respectively, and then add together the two solutions. This is another use of the principle of superposition.
PROBLEMS

Consider an elastic string of length $L$ whose ends are held fixed. The string is set in motion with no initial velocity from an initial position $u(x, 0) = f(x)$. In each of Problems 1 through 4 carry out the following steps. Let $L = 10$ and $a = 1$ in parts (b) through (d).

(a) Find the displacement $u(x, t)$ for the given initial position $f(x)$.

(b) Plot $u(x, t)$ versus $x$ for $0 \leq x \leq 10$ and for several values of $t$ between $t = 0$ and $t = 20$.

(c) Plot $u(x, t)$ versus $t$ for $0 \leq t \leq 20$ and for several values of $x$.

(d) Construct an animation of the solution in time for at least one period.

(e) Describe the motion of the string in a few sentences.

1. $f(x) = \begin{cases} 2x/L, & 0 \leq x \leq L/2, \\ 2(L-x)/L, & L/2 < x \leq L \end{cases}$

2. $f(x) = \begin{cases} 4x/L, & 0 \leq x \leq L/4, \\ 1, & L/4 < x < 3L/4, \\ 4(L-x)/L, & 3L/4 \leq x \leq L \end{cases}$

3. $f(x) = 8x(L-x)^2/L^3$

4. $f(x) = \begin{cases} 1, & L/2 - 1 < x < L/2 + 1 \ (L > 2), \\ 0, & \text{otherwise} \end{cases}$

Consider an elastic string of length $L$ whose ends are held fixed. The string is set in motion from its equilibrium position with an initial velocity $u_t(x, 0) = g(x)$. In each of Problems 5 through 8 carry out the following steps. Let
Find the displacement $u(x, t)$ for the given $g(x)$.

Plot $u(x, t)$ versus $x$ for $0 \leq x \leq 10$ and for several values of $t$ between $t = 0$ and $t = 20$.

Plot $u(x, t)$ versus $t$ for $0 \leq t \leq 20$ and for several values of $x$.

Construct an animation of the solution in time for at least one period.

Describe the motion of the string in a few sentences.

5. $g(x) = \begin{cases} 
2x/L, & 0 \leq x \leq L/2, \\
2(L - x)/L, & L/2 < x \leq L 
\end{cases}$

6. $g(x) = \begin{cases} 
4x/L, & 0 \leq x \leq L/4, \\
1, & L/4 < x < 3L/4, \\
4(L - x)/L, & 3L/4 \leq x \leq L 
\end{cases}$

7. $g(x) = 8x(L - x)^2/L^3$

8. $g(x) = \begin{cases} 
1, & L/2 - 1 < x < L/2 + 1 (L > 2), \\
0, & \text{otherwise} 
\end{cases}$

9. If an elastic string is free at one end, the boundary condition to be satisfied there is that $u_x = 0$. Find the displacement $u(x, t)$ in an elastic string of length $L$, fixed at $x = 0$ and free at $x = L$, set in motion with no initial velocity from the initial position $u(x, 0) = f(x)$, where $f$ is a given function.

Hint: Show that the fundamental solutions for this problem, satisfying all conditions except the nonhomogeneous initial condition, are

$$u_n(x, t) = \sin \lambda_n x \cos \lambda_n at,$$
where $\lambda_n = (2n - 1)\pi/2L$, $n = 1, 2, \ldots$. Compare this problem with Problem 15 of Section 10.6; pay particular attention to the extension of the initial data out of the original interval $[0, L]$.

10. Consider an elastic string of length $L$. The end $x = 0$ is fixed, while the end $x = L$ is free; thus the boundary conditions are $u(0, t) = 0$ and $u_x(L, t) = 0$. The string is set in motion with no initial velocity from the initial position $u(x, 0) = f(x)$, where

$$f(x) = \begin{cases} 1, & L/2 - 1 < x < L/2 + 1 (L > 2), \\ 0, & \text{otherwise}. \end{cases}$$

(a) Find the displacement $u(x, t)$.

(b) With $L = 10$ and $a = 1$, plot $u$ versus $x$ for $0 \leq x \leq 10$ and for several values of $t$. Pay particular attention to values of $t$ between 3 and 7. Observe how the initial disturbance is reflected at each end of the string.

(c) With $L = 10$ and $a = 1$, plot $u$ versus $t$ for several values of $x$.

(d) Construct an animation of the solution in time for at least one period.

(e) Describe the motion of the string in a few sentences.

11. Suppose that the string in Problem 10 is started instead from the initial position $f(x) = 8x(L - x)^2/L^3$. Follow the instructions in Problem 10 for this new problem.

12. Dimensionless variables can be introduced into the wave equation $a^2 u_{xx} = u_{tt}$ in the following manner. Let $s = x/L$ and show that the wave equation becomes

$$a^2 u_{ss} = L^2 u_{tt}.$$ 

Then show that $L/a$ has the dimensions of time and thus can be used as the unit on the time scale. Finally, let $\tau = at/L$ and show that the wave equation then reduces to

$$u_{ss} = u_{\tau\tau}.$$
Problems 13 and 14 indicate the form of the general solution of the wave equation and the physical significance of the constant \( a \).

13. Show that the wave equation

\[ a^2 u_{xx} = u_{tt} \]

can be reduced to the form \( u_{\xi \eta} = 0 \) by the change of variables \( \xi = x - at \), \( \eta = x + at \). Show that \( u(x, t) \) can be written as

\[ u(x, t) = \phi(x - at) + \psi(x + at) \]

where \( \phi \) and \( \psi \) are arbitrary functions.

Just Ask!

14. Plot the value of \( \phi(x - at) \) for \( t = 0 \), \( 1/a \), \( 2/a \), and \( t \neq 0 \) if \( \phi(s) = \sin s \). Note that for any \( t \neq 0 \) the graph of \( y = \phi(x - at) \) is the same as that of \( y = \phi(x) \) when \( t = 0 \), but displaced a distance \( at \) in the positive \( x \) direction. Thus \( a \) represents the velocity at which a disturbance moves along the string. What is the interpretation of \( \phi(x + at) \)?

15. A steel wire 5 ft in length is stretched by a tensile force of 50 lb. The wire has a weight per unit length of 0.026 lb/ft.

(a) Find the velocity of propagation of transverse waves in the wire.

(b) Find the natural frequencies of vibration.

(c) If the tension in the wire is increased, how are the natural frequencies changed? Are the natural modes also changed?

16. Consider the wave equation

\[ a^2 u_{xx} = u_{tt} \]

in an infinite one-dimensional medium subject to the initial conditions

\[ u(x, 0) = f(x), \quad u_t(x, 0) = 0, \quad -\infty < x < \infty. \]

(a) Using the form of the solution obtained in Problem 13, show that \( \phi \) and \( \psi \) must satisfy
\[ \phi(x) + \psi(x) = f(x), \]
\[ -\phi'(x) + \psi'(x) = 0. \]

(b) Solve the equations of part (a) for \( \phi \) and \( \psi \), and thereby show that
\[ u(x, t) = \frac{1}{2} \left[ f(x - at) + f(x + at) \right]. \]

This form of the solution was obtained by D'Alembert in 1746.

Hint: Note that the equation \( \psi'(x) = \phi'(x) \) is solved by choosing \( \psi(x) = \phi(x) + c \).

(c) Let
\[
f(x) = \begin{cases} 
2, & -1 < x < 1, \\
0, & \text{otherwise}.
\end{cases}
\]

Show that
\[
f(x - at) = \begin{cases} 
2, & -1 + at < x < 1 + at, \\
0, & \text{otherwise}.
\end{cases}
\]

Also determine \( f(x + at) \).

(d) Sketch the solution found in part (b) at \( t = 0, t = 1/2a, t = 1/a \), and \( t = 2/a \), obtaining the results shown in Figure 10.7.7. Observe that an initial displacement produces two waves moving in opposite directions away from the original location; each wave consists of one-half of the initial displacement.
17. Consider the wave equation

\[ a^2 u_{xx} = u_{tt} \]

in an infinite one-dimensional medium subject to the initial conditions

\[ u(x,0) = 0, \quad u_t(x,0) = g(x), \quad -\infty < x < \infty. \]

(a) Using the form of the solution obtained in Problem 13, show that

\[ \phi(x) + \psi(x) = 0, \]
\[ -a \phi'(x) + a \psi'(x) = g(x). \]

(b) Use the first equation of part (a) to show that \( \psi'(x) = -\phi'(x) \). Then use the second equation to show that \( -2a \phi'(x) = g(x) \) and therefore that
\[
\phi(x) = -\frac{1}{2a} \int_{x_0}^{x} g(\xi) d\xi + \phi(x_0),
\]
where \(x_0\) is arbitrary. Finally, determine \(\psi(x)\).

(c) Show that
\[
u(x, t) = \frac{1}{2a} \int_{x-at}^{x+at} g(\xi) d\xi.
\]

Just Ask!

18. By combining the results of Problems 16 and 17, show that the solution of the problem
\[
a^2 u_{xx} = u_{tt},
u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad -\infty < x < \infty
\]
is given by
\[
u(x, t) = \frac{1}{2} \left[ f(x - at) + f(x + at) \right] + \frac{1}{2a} \int_{x-at}^{x+at} g(\xi) d\xi.
\]

Problems 19 and 20 indicate how the formal solution (20), (22) of Eqs. (1), (3), and (9) can be shown to constitute the actual solution of that problem.

19. By using the trigonometric identity \(\sin A \cos B = \frac{1}{2} [\sin(A + B) + \sin(A - B)]\) show that the solution (20) of the problem of Eqs. (1), (3), and (9) can be written in the form (28).

20. Let \(h(\xi)\) represent the initial displacement in \([0, L]\), extended into \((-L, 0)\) as an odd function and extended elsewhere as a periodic function of period \(2L\). Assuming that \(h, h', h''\) are continuous, show by direct differentiation that \(u(x, t)\) as given in Eq. (28) satisfies the wave equation (1) and also the initial conditions (9). Note also that since Eq. (20) clearly satisfies the boundary conditions (3), the same is true of Eq. (28). Comparing Eq. (28) with the solution of the corresponding problem for the infinite string (Problem 16), we see that they have the same form, provided that the initial data for the finite string, defined originally only on the interval \(0 \leq x \leq L\), are extended in the given manner over the entire \(x\)-axis. If this is done, the solution for the infinite string is also applicable to the finite one.

21. The motion of a circular elastic membrane, such as a drumhead, is governed by the two-dimensional wave equation in polar coordinates...
22. The total energy $E(t)$ of the vibrating string is given as a function of time by

$$E(t) = \int_0^L \left[ \frac{1}{2} \rho u_t^2(x, t) + \frac{1}{2} T u_x^2(x, t) \right] dx; \quad (i)$$

the first term is the kinetic energy due to the motion of the string, and the second term is the potential energy created by the displacement of the string away from its equilibrium position.

For the displacement $u(x, t)$ given by Eq. (20) —that is, for the solution of the string problem with zero initial velocity—show that

$$E(t) = \frac{\pi^2 T}{4L} \sum_{n=1}^{\infty} n^2 c_n^2. \quad (ii)$$

Note that the right side of Eq. (ii) does not depend on $t$. Thus the total energy $E$ is a constant and therefore is conserved during the motion of the string.

*Hint:* Use Parseval’s equation (Problem 37 of Section 10.4 and Problem 17 of Section 10.3), and recall that $a^2 = T/\rho$.

23. **Dispersive Waves.** Consider the modified wave equation

$$a^{-2} u_{tt} + \gamma^2 u = u_{xx}, \quad 0 < x < L, \quad t > 0 \quad (i)$$

with the boundary conditions

$$u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0 \quad (ii)$$

and the initial conditions

$$u(x, 0) = f(x), \quad u_t(x, 0) = 0, \quad 0 < x < L. \quad (iii)$$

(a) Show that the solution can be written as
where
\[ c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx. \]

(b) By using trigonometric identities, rewrite the solution as
\[ u(x, t) = \frac{1}{2} \sum_{n=1}^{\infty} c_n \left[ \sin \frac{n\pi x}{L} (x + a_nt) + \sin \frac{n\pi x}{L} (x - a_nt) \right]. \]

Determine \(a_n\), the speed of wave propagation.

(c) Observe that \(a_n\), found in part (b), depends on \(n\). This means that components of different wave lengths (or frequencies) are propagated at different speeds, resulting in a distortion of the original wave form over time. This phenomenon is called dispersion. Find the condition under which \(a_n\) is independent of \(n\) — that is, there is no dispersion.

24. Consider the situation in Problem 23 with \(a^2 = 1, L = 10\). and

\[ f(x) = \begin{cases} 
    x - 4, & 4 \leq x \leq 5, \\
    6 - x, & 5 \leq x \leq 6, \\
    0, & \text{otherwise}.
\end{cases} \]

(a) Determine the coefficients \(c_n\) in the solution of Problem 23(a). By plotting \(\sum_{n=1}^{N} c_n \sin \frac{n\pi x}{10}\) for \(0 \leq x \leq 10\), choose \(N\) large enough so that the plot accurately displays the graph of \(f(x)\). Use this value of \(N\) for the remaining plots called for in this problem.

(b) Let \(\gamma = 0\). Plot \(u(x, t)\) versus \(x\) for \(t = 60\).

(c) Let \(\gamma = 1/8\). Plot \(u(x, t)\) versus \(x\) for \(t = 20\), \(40\), and \(60\).

(d) Let \(\gamma = 1/4\). Plot \(u(x, t)\) versus \(x\) for \(t = 20\), \(40\), and \(60\).