3.3: WAVE EQUATION AND THE METHOD OF SEPARATION OF VARIABLES

KIAM HEONG KWA

The goal of this section is to present the method of separation of variables as an approach to solve the one-dimensional wave equation
\begin{equation}
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial u}{\partial x}, \quad 0 < x < L, \ t > 0,
\end{equation}
where \( c > 0 \), subject to the boundary conditions
\begin{equation}
 u(0, t) = u(L, t) = 0 \text{ for all } t > 0
\end{equation}
and the initial conditions
\begin{equation}
 u(x, 0) = f(x) \text{ and } \frac{\partial u}{\partial t}(x, 0) = g(x) \text{ for } 0 < x < L.
\end{equation}

Step 1: Separating Variables in (1.1) and (1.2). We begin by searching for nonzero product solutions\(^1\) of (1.1) of the form
\begin{equation}
 u(x, t) = X(x)T(t),
\end{equation}
where \( X \) is a function of \( x \) alone and \( T \) is a function of \( t \) alone. It is worth noting that
\begin{equation}
 X(x) \neq 0 \text{ and } T(t) \neq 0
\end{equation}
because we have assumed that such a product solution \( u = XT \) is nonzero. Differentiating (1.4) with respect to \( t \) and \( x \) yields
\begin{equation}
 \frac{\partial^2 u}{\partial t^2} = X(x)T''(t) \text{ and } \frac{\partial^2 u}{\partial x^2} = X''(x)T(t).
\end{equation}
Plugging these into (1.1) and dividing the resulting equation by \( c^2XT \) gives
\begin{equation}
 \frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)}.
\end{equation}

\(^1\)As we will see soon, these product solutions are building blocks of more general solutions. \textit{We cannot} really build any solutions from the trivial one, whence the assumption that the product solutions are nonzero.
This way we have **separated** the variables in the sense that the left side of (1.7) is a function of \( t \) alone, while the right side is a function of \( x \) alone. Since \( t \) and \( x \) are variables independent of each other, this can occur only if both sides of (1.7) are constant and equal\(^2\):

\[
(1.8) \quad \frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = k.
\]

Such a constant \( k \) is called a **separation constant**. Note that (1.8) holds if and only if so do

\[
(1.9) \quad X'' - kX = 0
\]

and

\[
(1.10) \quad T'' - kc^2T = 0.
\]

So far we have separated the variables \( t \) and \( x \) in (1.1).

We can also separate the variables in the boundary conditions (1.2). By (1.2) and (1.4), we have

\[
(1.11) \quad X(0)T(t) = X(L)T(t) = 0 \text{ for all } t > 0.
\]

If \( X(0) \neq 0 \) or \( X(L) \neq 0 \), then \( T(t) = 0 \) for all \( t > 0 \). This will not consist with (1.5). Hence we set

\[
(1.12) \quad X(0) = X(L) = 0.
\]

**Step 2: Solving the Separated Equations** (1.9) and (1.10). Combining (1.9) and (1.12), we arrive at a boundary value problem in \( X \):

\[
(1.13) \quad X'' - kX = 0, \quad X(0) = X(L) = 0.
\]

It should be noted that not all values of the separation constant \( k \) yield nontrivial solutions for (1.13).

If \( k > 0 \), so that \( k = \mu^2 \) with \( \mu > 0 \), then

\[
X'' - \mu^2X = 0,
\]

from which it follows that

\[
X(x) = c_1 \cosh \mu x + c_2 \sinh \mu x,
\]

\(^2\)Let \( f(t) = g(x) \), where \( f \) is a function of \( t \) alone and \( g \) is a function of \( x \) alone. Then \( \frac{df}{dt} = \frac{\partial f}{\partial t} = \frac{\partial g}{\partial t} = 0 \) and \( \frac{dg}{dx} = \frac{\partial g}{\partial x} = \frac{\partial f}{\partial x} = 0 \), from which it follows that \( f \) and \( g \) are constant and equal.
where \( c_1 \) and \( c_2 \) are integration constants. Then the boundary conditions \( X(0) = X(L) = 0 \) imply that

\[
\begin{align*}
  c_1 &= c_1 \cosh 0 + c_2 \sinh 0 = X(0) = 0, \\
  c_2 \sinh \mu L &= c_1 \cosh \mu L + c_2 \sinh \mu L = X(L) = 0.
\end{align*}
\]

This in turn implies that \( c_1 = c_2 = 0 \), so that \( X = 0 \) if \( k > 0 \).

If \( k = 0 \), then

\[
X'' = 0,
\]

whence it follows that

\[
X(x) = c_1 x + c_2,
\]

where \( c_1 \) and \( c_2 \) are integration constants. The boundary conditions \( X(0) = X(L) = 0 \) imply that

\[
\begin{align*}
  c_2 &= c_1 \cdot 0 + c_2 = X(0) = 0, \\
  c_1 &= c_1 L + c_2 = X(L) = 0.
\end{align*}
\]

So, \( X = 0 \) if \( k = 0 \).

Finally, suppose that \( k < 0 \), so that \( k = -\mu^2 \) with \( \mu > 0 \). Then

\[
X'' + \mu^2 = 0.
\]

The general solution of this equation is

\[
X(x) = c_1 \cos \mu x + c_2 \sin \mu x,
\]

where \( c_1 \) and \( c_2 \) are integration constants. The boundary conditions \( X(0) = X(L) = 0 \) imply that

\[
\begin{align*}
  c_1 &= c_1 \cos 0 + c_2 \sin 0 = X(0) = 0, \\
  c_2 \sin \mu L &= c_1 \cos \mu L + c_2 \sin \mu L = X(L) = 0.
\end{align*}
\]

Clearly, \( c_1 = 0 \), so that \( X(x) = c_2 \sin \mu x \). Also, from the boundary condition \( X(L) = 0 \), we have either \( c_2 = 0 \) or \( \sin \mu L = 0 \). If \( c_2 = 0 \), then \( X = 0 \). To avoid trivializing \( X \), we take \( c_2 = 1 \) (or any nonzero value of \( c_2 \)) and \( \sin \mu L = 0 \), so that

\[
\mu L = \mu_n L = n\pi, \ n = 1, 2, \cdots,
\]

or equivalently

\[
\mu = \mu_n = \frac{n\pi}{L}, \ n = 1, 2, \cdots,
\]

from which it follows that

\[
k = k_n = -\mu_n^2 = -\frac{n^2\pi^2}{L^2}, \ n = 1, 2, \cdots,
\]
are the only values of the separation constant for which (1.13) admits nontrivial solutions. In addition, for each of these values \( k = k_n \),

\[
X(x) = X_n(x) = \sin \mu_n x = \sin \frac{n\pi x}{L}
\]

is such a nontrivial solution.

For each \( n, n = 1, 2, \cdots \), substituting \( k \) by \( k_n = -\mu_n^2 \) in (1.10) yields

\[
T'' + c^2 \mu_n^2 T = 0,
\]

the general solution of which has the form

\[
T_n(t) = b_n \cos c\mu_n t + b_n^* \sin c\mu_n t = b_n \cos \frac{cn\pi t}{L} + b_n^* \sin \frac{cn\pi t}{L},
\]

where \( b_n \) and \( b_n^* \) are integration constants.

Combining (1.15) and (1.16) yields the following infinite set of product solutions of (1.1) subject to the boundary conditions (1.2):

\[
u_n(x, t) = X_n(x)T_n(t) = \sin \frac{n\pi x}{L} \left( b_n \cos \frac{cn\pi t}{L} + b_n^* \sin \frac{cn\pi t}{L} \right),
\]

These are called the **normal modes** of the wave equation.

**Step 3: Fourier Series Solution of the Entire Problem.** In order to take care of the general initial conditions (1.3), we form the series solution

\[
u(x, t) = \sum_{n=1}^{\infty} \nu_n(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left( b_n \cos \frac{cn\pi t}{L} + b_n^* \sin \frac{cn\pi t}{L} \right)
\]

from the normal modes (1.17). Differentiating this series solution term by term with respect to \( t \) yields

\[
\frac{\partial \nu}{\partial t} = \sum_{n=1}^{\infty} \frac{cn\pi}{L} \sin \frac{n\pi x}{L} \left( -b_n \sin \frac{cn\pi t}{L} + b_n^* \cos \frac{cn\pi t}{L} \right).
\]

Then the initial conditions (1.3) demand that

\[
f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}
\]

and

\[
g(x) = \sum_{n=1}^{\infty} \frac{cn\pi b_n^*}{L} \sin \frac{n\pi x}{L}.
\]
These equations can be fulfilled by setting

\[ b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx \]

and

\[ b_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx \]

for \( n = 1, 2, \cdots \); that is, by setting \( b_n \) and \( \frac{cn\pi b_n}{L} \) as the coefficients in the Fourier sine series of \( f \) and \( g \) respectively.

**Example 1** (Exercise 3.3.2(a) in the text). With \( L = 1, \ c = \frac{1}{\pi} \), and the initial data

\[ f(x) = \sin \pi x \cos \pi x = \frac{1}{2} \sin 2\pi x, \ g(x) = 0, \ \text{and} \ c = \frac{1}{\pi}, \]

the series solution (1.18) of the wave equation needs to satisfy the boundary conditions

\[ u(x, 0) = \sum_{n=1}^{\infty} b_n \sin n\pi x = \frac{1}{2} \sin 2\pi x \]

and

\[ \frac{\partial u}{\partial t}(x, 0) = \sum_{n=1}^{\infty} nb_n^* \sin n\pi x = 0. \]

These can be achieved by setting

\[ b_n = \begin{cases} 
\frac{1}{2} & \text{if } n = 2, \\
0 & \text{otherwise; and } b_n^* = 0 \text{ for all } n.
\end{cases} \]

So the solution of the wave equation is

\[ u(x, t) = \frac{1}{2} \sin 2\pi x \cos 2t. \]

**Example 2** (Exercise 3.3.4(a) in the text). With \( L = 1, \ c = 1, \) and the initial data

\[ f(x) = \sin \pi x + \frac{1}{2} \sin 3\pi x + 3 \sin 7\pi x, \ g(x) = \sin 2\pi x, \]
the series solution (1.18) of the wave solution needs to satisfy the boundary conditions

\[ u(x, 0) = \sum_{n=1}^{\infty} b_n \sin n\pi x = \sin \pi x + \frac{1}{2} \sin 3\pi x + 3 \sin 7\pi x \]

and

\[ \frac{\partial u}{\partial t}(x, 0) = \sum_{n=1}^{\infty} n\pi b_n^* \sin n\pi x = \sin 2\pi x. \]

This can be achieved by setting

\[ b_n = \begin{cases} 1 & \text{if } n = 1, \\ \frac{1}{2} & \text{if } n = 3, \\ 3 & \text{if } n = 7, \\ 0 & \text{otherwise}; \end{cases} \]

and

\[ n\pi b_n^* = \begin{cases} 1 & \text{if } n = 2, \\ 0 & \text{otherwise}. \end{cases} \]

So the solution of the wave equation is

\[ u(x, t) = \sin \pi x \cos \pi t + \frac{1}{2\pi} \sin 2\pi x \sin 2\pi t + \frac{1}{2} \sin 3\pi x \cos 3\pi t + 3 \sin 7\pi x \cos 7\pi t. \]

Example 3 (Exercise 3.3.6(a) in the text). With \( L = 1 \), \( c = \frac{1}{\pi} \), and the initial data

\[ f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{3}, \\ \frac{1}{30} (x - \frac{1}{3}) & \text{if } \frac{1}{3} \leq x \leq \frac{2}{3}, \text{ and } g(x) = 2, \\ \frac{1}{30} (1 - x) & \text{if } \frac{2}{3} < x \leq 1; \end{cases} \]

the series solution (1.18) of the wave solution needs to satisfy the boundary conditions

\[ u(x, 0) = \sum_{n=1}^{\infty} b_n \sin n\pi x = f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{3}, \\ \frac{1}{30} (x - \frac{1}{3}) & \text{if } \frac{1}{3} \leq x \leq \frac{2}{3}, \\ \frac{1}{30} (1 - x) & \text{if } \frac{2}{3} < x \leq 1 \end{cases} \]

and

\[ \frac{\partial u}{\partial t}(x, 0) = \sum_{n=1}^{\infty} nb_n^* \sin n\pi x = g(x) = 2. \]
This can be achieved by setting
\[
b_n = 2 \int_0^1 f(x) \sin n\pi x \, dx
\]
\[
= 2 \int_0^{1/3} 0 \cdot \sin n\pi x \, dx + 2 \int_{1/3}^{2/3} \frac{1}{30} \left( x - \frac{1}{3} \right) \sin n\pi x \, dx
\]
\[
+ 2 \int_{2/3}^1 \frac{1}{30} (1 - x) \sin n\pi x \, dx
\]
\[
= 2 \left[ \left( x - \frac{1}{3} \right) \cdot \frac{\cos n\pi x}{n\pi} - 1 \cdot \frac{n\pi}{n^2\pi^2} \right]_{1/3}^{2/3}
\]
\[
+ \frac{2}{30} \left[ (1 - x) \cdot \frac{\cos n\pi x}{n\pi} - (-1) \cdot \frac{\sin n\pi x}{n^2\pi^2} \right]_{1/3}^{2/3}
\]
\[
= \frac{1}{15n^2\pi^2} \left( 2 \sin \frac{2n\pi}{3} - \sin \frac{n\pi}{3} \right)
\]

and
\[
b_n' = \frac{1}{n} \int_0^1 g(x) \sin n\pi x \, dx
\]
\[
= \frac{4}{n^2\pi} \left[ 1 - (-1)^n \right]
\]
\[
= \begin{cases} 
\frac{8}{n^2\pi} & \text{if } n \text{ is odd}, \\
0 & \text{if } n \text{ is even}
\end{cases}
\]

for \( n = 1, 2, \cdots \). So the solution of the wave equation is
\[
u(x, t) = \sum_{n=1}^{\infty} \sin n\pi x \left\{ \frac{1}{15n^2\pi^2} \left( 2 \sin \frac{2n\pi}{3} - \sin \frac{n\pi}{3} \right) \cos nt \\
+ \frac{4}{n^2\pi} \left[ 1 - (-1)^n \right] \sin nt \right\}
\]
\[
= \sum_{n=1}^{\infty} \sin n\pi x \left\{ \frac{1}{15\pi} \left( 2 \sin \frac{2n\pi}{3} - \sin \frac{n\pi}{3} \right) \cos nt + 4[1 - (-1)^n] \sin nt \right\}.
\]